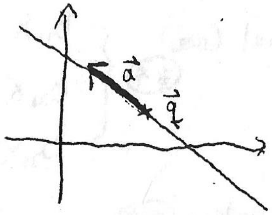
Summary: Parametric and Equation of Lines in $\mathbb{R}^2$

Given a point $\vec{q}$ on a straight line L and a direction $\vec{a}$ of L



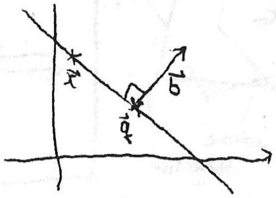
Parametric form of the line is

$$\vec{x} = \vec{q} + s\vec{a}$$

i.e. $[x_1, x_2] = [q_1 + sa_1, q_2 + sa_2]$

Given a point $\vec{q}$ on a straight line L and a normal direction $\vec{b}$ of L

For any point $\vec{x} = [x_1, x_2]$ on L



$$(\vec{x} - \vec{q}) \cdot \vec{b} = 0$$

$$[x_1 - q_1, x_2 - q_2] \cdot [b_1, b_2] = 0$$

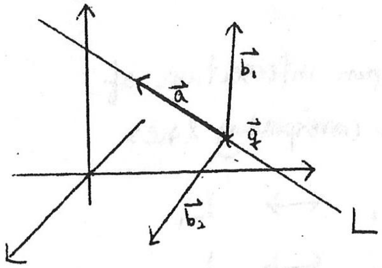
$$b_1(x_1 - q_1) + b_2(x_2 - q_2) = 0$$

∴ Equation form of L is

$$b_1x_1 + b_2x_2 = c \text{ , where } c = \vec{b} \cdot \vec{q} = b_1q_1 + b_2q_2$$

Summary: Parametric and Equation form of Lines and Planes in $\mathbb{R}^3$

Line L



- $\vec{q} = [q_1, q_2, q_3]$ is a point on L
- $\vec{a} = [a_1, a_2, a_3]$ is a direction of L
- $\vec{b_1} = [b_{11}, b_{12}, b_{13}]$, $\vec{b_2} = [b_{21}, b_{22}, b_{23}]$ are two (non-zero) non-collinear vectors orthogonal (perpendicular) to L

To describe points $\vec{x} = [x_1, x_2, x_3]$ on L,

Parametric form

$$\vec{x} = \vec{q} + s\vec{a}$$

i.e. $[x_1, x_2, x_3] = [q_1 + sa_1, q_2 + sa_2, q_3 + sa_3]$

Equation form

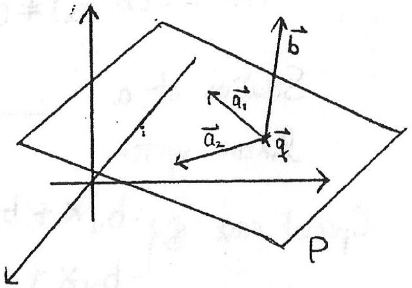
$$\begin{cases} (\vec{x} - \vec{q}) \cdot \vec{b_1} = 0 \\ (\vec{x} - \vec{q}) \cdot \vec{b_2} = 0 \end{cases}$$

i.e.

$$\begin{cases} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 = c_1 \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 = c_2 \end{cases}$$

where $c_1 = \vec{b_1} \cdot \vec{q}$, $c_2 = \vec{b_2} \cdot \vec{q}$

Plane P



- $\vec{q} = [q_1, q_2, q_3]$ is a point on P
- $\vec{a_1} = [a_{11}, a_{12}, a_{13}]$, $\vec{a_2} = [a_{21}, a_{22}, a_{23}]$ are two (non-zero) non-collinear vectors in the direction of P
- $\vec{b} = [b_1, b_2, b_3]$ is orthogonal to L

To describe points $\vec{x} = [x_1, x_2, x_3]$ on P,

Parametric form

$$\vec{x} = \vec{q} + s\vec{a_1} + t\vec{a_2}$$

i.e. $[x_1, x_2, x_3]$

$= [q_1 + sa_{11} + ta_{21}, q_2 + sa_{12} + ta_{22}, q_3 + sa_{13} + ta_{23}]$

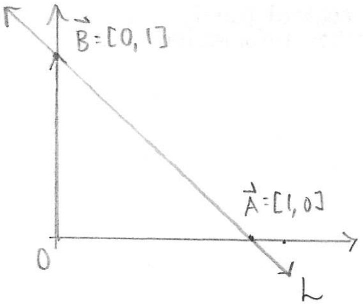
Equation form

$$(\vec{x} - \vec{q}) \cdot \vec{b} = 0$$

i.e. $b_1x_1 + b_2x_2 + b_3x_3 = c$

where $c = \vec{b} \cdot \vec{q}$

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$



Equation Form

$$x+y=1$$

Parametric Form

$$[x,y] = [1,0] + \underset{\substack{\uparrow \\ \text{parameter}}}{s}\underbrace{[-1,1]}_{\text{direction}}$$

If $s=0$, $[x,y] = [1,0]$
$\quad =1 \qquad = [0,1]$
$\quad =2 \qquad = [-1,2]$
$\quad =-1 \qquad = [2,-1]$

By varying s, we get all points on L

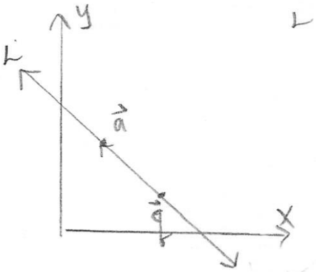
Parametric Form of a Line expressed in different ways:

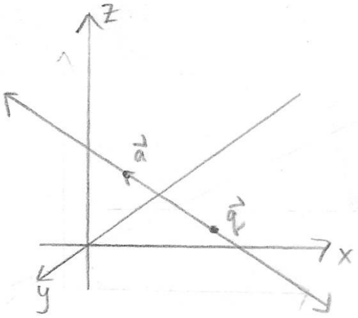
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1+3s \\ 2+4s \end{bmatrix}$
- $\begin{Bmatrix} x = 1+3s \\ y = 2+4s \end{Bmatrix}$

- Parametric Form:
  - a point $\vec{q}$ on L
  - a direction $\vec{a}$ of L

$$\vec{x} = [\overset{x}{x_1} \quad \overset{y}{x_2}] = \vec{q} + s\vec{a}$$

$$\vec{x} = [\overset{x}{x_1} \quad \overset{y}{x_2} \quad \overset{z}{x_3}] = \vec{q} + s\vec{a}$$





- Equation Form ($\mathbb{R}^2$)
  - a point $\vec{q}$ on L
  - a normal direction $\vec{b}$ of L

$\vec{x} = [x, y]$ on L

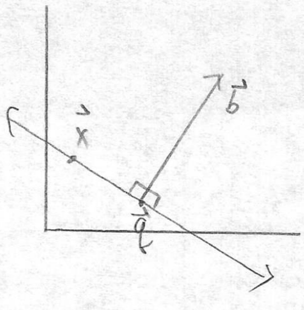
$$(\vec{x}-\vec{q}) \cdot \vec{b} = 0$$

$$[x-q_1, y-q_2] \cdot [b_1, b_2]$$

$$b_1(x-q_1) + b_2(y-q_2) = 0$$

$$b_1 x + b_2 y = \underbrace{b_1 q_1 + b_2 q_2}_{\text{constant}}$$

$$b_1 x + b_2 y = \vec{b} \cdot \vec{q}$$



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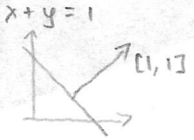
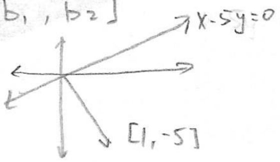
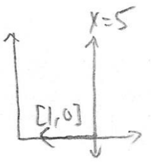
① The line $b_1x + b_2y = c$ is $\perp$ to $[b_1, b_2]$







② $[b_1, b_2] \perp [-b_2, b_1]$

- Equation Form ($\mathbb{R}^3$)
  - a point $\vec{q}$ on L
  - TWO non-zero, non colinear normal directions (vectors orthogonal) $\vec{b_1}$ and $\vec{b_2}$ of L

$\vec{x} = [x, y, z]$ on L

- $(\vec{x}-\vec{q}) \cdot \vec{b_1} = 0$

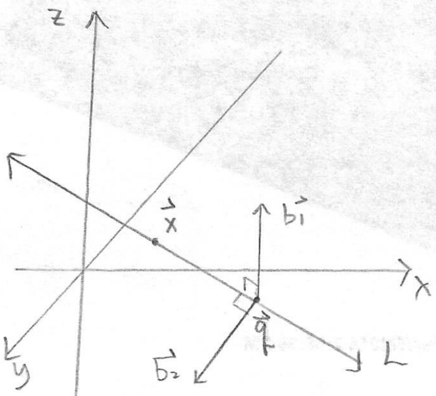
$$b_{11}x + b_{12}y + b_{13}z = \vec{b_1} \cdot \vec{q} = C_1$$

- $(\vec{x}-\vec{q}) \cdot \vec{b_2} = 0$

$$b_{21}x + b_{22}y + b_{23}z = \vec{b_2} \cdot \vec{q} = C_2$$

$$b_{11}x + b_{12}y + b_{13}z = \vec{b_1} \cdot \vec{q}$$
$$b_{21}x + b_{22}y + b_{23}z = \vec{b_2} \cdot \vec{q}$$

Need 2 eqn's to describe the line, the intersection of 2 planes.



Describing a Line in $\mathbb{R}^2$

Suppose $P = [1, 1]$ and $Q = [3, 0]$ are 2 points on a straight line in $\mathbb{R}^2$.

Parametric: Direction $= \vec{PQ} = \vec{Q} - \vec{P} = [3, 0] - [1, 1] = [2, -1]$

$$[x, y] = [1, 1] + s[2, -1]$$

$$= \begin{cases} x = 2s + 1 \\ y = -s + 1 \end{cases}$$

Equation: Need a non-zero normal direction of L.

A direction of L is $[2, -1]$

$[2, -1] \perp [1, 2] \Rightarrow [1, 2]$ is a normal direction

or choose any non-zero vector where dot product $= 0$

$\vec{x} = [x, y]$ on L

$$(\vec{x} - \vec{q}) \cdot \vec{b} = 0$$

$$[x-1, y-1] \cdot [1, 2] = 0$$

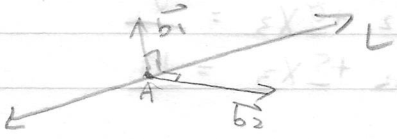
$$x - 1 + 2(y-1) = 0.$$

$$x + 2y = 3$$

Describing a Line in $\mathbb{R}^3$

Jan 19 II

To describe a line in 3D, you need 2 eq'ns (for equation form)
The intersection of these 2 eq'ns (planes) defines the line.



e.g. Define a line in $\mathbb{R}^3$ that passes through $P = [0\ 0\ 1]$, $Q = [2\ 4\ 0]$

Parametric: Direction $\vec{PQ} = Q - P = [2, 4, -1]$

$[x\ y\ z] = [0, 0, 1] + s[2, 4, -1]$

Equation: Need 2 non-colinear normal vectors, both $\perp$ to $[2, 4, -1]$

$$[2, 4, -1] \cdot [1, 0, 2] = 0$$
$$[2, 4, -1] \cdot [-4, 2, 0] = 0$$

$b_1 = [1\ \ 0\ \ 2]$ $\qquad$ $b_2 = [-4, 2, 0]$

$(\vec{v} - \vec{p}) \cdot \vec{b_1} = 0$ $\qquad$ $(\vec{v} - \vec{p}) \cdot \vec{b_2} = 0$

$([x\ y\ z] - [0\ 0\ 1]) \cdot \vec{b_1} = 0$ $\qquad$ $([x\ y\ z] - [0\ 0\ 1]) \cdot \vec{b_2} = 0$

$[x, y, z-1] \cdot [1, 0, 2] = 0$ $\qquad$ $[x, y, z-1] \cdot [-4, 2, 0] = 0$

$x + 2(z-1) = 0$ $\qquad$ $-4x + 2y = 0$

$x + 2z = 2$ $\qquad$ $2x - y = 0$

WW 2.⑭ Find a pair of eq'ns which defines the line

$$x = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + t \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix}$$

$$\underline{1}\, x_1 + \underline{-1}\, x_2 + \underline{0}\, x_3 = \underline{5}$$
$$\underline{2}\, x_1 + \underline{0}\, x_2 + \underline{5}\, x_3 = \underline{19}$$

a direction: $[5, 5, -2]$

2 non-colinear normal directions

$[5, 5, -2] \cdot [1, -1, 0] = 0$ $\quad$ $[5, 5, -2] \cdot [2, 0, 5] = 0$

$(\vec{p} - [2, -3, 3]) \cdot [1, -1, 0] = 0$

$[x - 2, y + 3, z - 3] \cdot [1, -1, 0] = 0$ $\quad$ $[x-2, y+3, z-3] \cdot [2, 0, 5] = 0$

$x - y = 5$ $\quad$ $2x + 5z = 19$

WW (2.7) Consider the line $L(t) = \begin{bmatrix} 2+t \\ 3+2t \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2+t \\ 3+2t \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leftarrow \text{a direction of } L$$

L is __ (parallel, perpendicular, or neither) to

a) $\begin{bmatrix} -5-2t \\ -5-4t \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} - 2t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ parallel $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

b) $\begin{bmatrix} 1.5t-3 \\ 3t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 1.5t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ parallel $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

* c) $\begin{bmatrix} 1-4t \\ 1+2t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $(b_1, b_2) \perp (-b_2, b_1)$

$(-2,1) \cdot (1,2) = 0; \perp$ $\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = -3 = \|a\| \|b\| \cos \phi$ $[-2, 1] \cdot [1, 2] = 0$

d) $\begin{bmatrix} -5-t \\ 3+4t \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ neither $\begin{vmatrix} -1 & 1 \\ 4 & 2 \end{vmatrix} = -6$ [-1.

e) $\begin{bmatrix} 4t-3 \\ 5-t \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ neither

Dot Product $= 0$ if $\perp$
Determinant $= 0$ if $\parallel$

WW (2.8) The first set of parametric equations

$$x_1 = -8t + 3$$

$$x_2 = 2t + 1$$

$$x_3 = 8t + 8$$

and the second set of parametric equations

$$x_1 = 4t$$

$$x_2 =$$

$$x_3 =$$

describe the same line in $\mathbb{R}^3$.

$$\begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} + t \underbrace{\begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}}_{\text{a direction}}$$

$$\begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} - t \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4t \\ ? \\ ? \end{bmatrix} \qquad s = 3/4$$

$$\begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} + 3/4 \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} - t \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4t \\ 1.75 - t \\ 11 - 4t \end{bmatrix}$$

$$x_1 = 4t$$

$$x_2 = 1.75 - t$$

$$x_3 = 11 - 4t$$

ww.(2.10) Find the point of intersection between

L_1 $\quad x = -t$ $\qquad\qquad$ L_2 $\quad x = 7 + 2s$

$\qquad y = 12 + 2t$ $\qquad\qquad\qquad y = 2 - 2s$

$\qquad z = 11 + 3t$ $\qquad\qquad\qquad z = 10 + 4s$

Note that in $\mathbb{R}^3$, 2 lines in general do not intersect, but these have been carefully chosen so they do.

$$-t = 7 + 2s$$
$$12 + 2t = 2 - 2s$$
$$11 + 3t = 10 + 4s$$

$2s + t = -7 \Rightarrow t = -7 - 2s$ $\qquad t = -7 - 2(-2) = -3$

$2t + 2s = -10$ $\qquad \Rightarrow 2(-7-2s) + 2s = -10$

$\qquad -2s = 4$

$\qquad s = -2$

$3t - 4s = -1$

$\Rightarrow 3(-3) - 4(-2) = -1$ ✓

$(s, t) = (-2, -3)$ $\qquad [3, 6, 2]$

Find the point whee the line $x = -3 + 2t$

$y = 4 + 2t$

$z = -3 + 3t$

intersects the plane $-x + 2y + z = 2$

$$-(-3+2t) + 2(4+2t) + (-3+3t) = 2$$
$$3 - 2t + 8 + 4t - 3 + 3t = 2$$
$$5t = -6$$
$$t = -6/5$$

$x = -3 + 2(-6/5)$ $\qquad y = 4 + 2(-6/5)$ $\qquad z = -3 + 3(-6/5)$

$= -3 - 12/5$ $\qquad = 1.6$ $\qquad = -3 - 18/5$

$= -5.4$ $\qquad\qquad = -6.6$

$[-5.4, 1.6, -6.6]$