Matrices

rectangular array of numbers

$m \times n$ matrix : m rows, n columns

$$\begin{array}{cc} & \overset{2}{\longleftrightarrow} \\ 3 & \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \end{array}$$

3x2 matrix

$$\begin{array}{cc} & \overset{4}{\longleftrightarrow} \\ 2 & \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 5 & 2 \end{bmatrix} \end{array}$$

2x4 matrix

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

4x1 matrix

column vector

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

1x4 matrix

row vector

$1 \times n$ matrix is a row vector in $\mathbb{R}^n$ $\quad [1 \ 2 \ 3]$

$m \times 1$ matrix is a column vector in $\mathbb{R}^m$ $\quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

If $m=n$, the matrix is a square matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \qquad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

2 x 2 $\qquad$ 3 x 3

not a diagonal

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

diagonal
top left to bottom right

Determinants

of 2x2 and 3x3 matrices

$\mathbb{R}^2$: $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \vec{a} \times \vec{b} = a_1 b_2 - a_2 b_1$

$\mathbb{R}^3$: $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

e.g. $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - (2)(3) = -2$

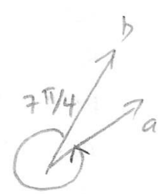
e.g. $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$$= (45-48) - 2(36-42) + 3(32-35)$$

Geometric Meaning

① see 'cross product'. '⊥' to both

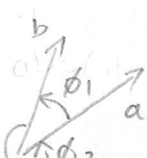
② $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \|a\| \, \|b\| \sin \emptyset$



$\emptyset$ is the angle, oriented, measured from $\vec{a}$ to $\vec{b}$ ccw.

If $\vec{a}, \vec{b} \neq \vec{0}$, $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{cases} >0 & \text{for } 0 < \emptyset < \pi \\ =0 & \text{for } \emptyset = 0 \text{ or } \pi \\ <0 & \text{for } \pi < \emptyset < 2\pi \end{cases}$

* better to use dot product for angle though



$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = |a| \, |b| \sin \emptyset_1$

$\det \begin{bmatrix} b_1 & b_2 \\ a_1 & a_2 \end{bmatrix} = |a| \, |b| \sin \emptyset_2$

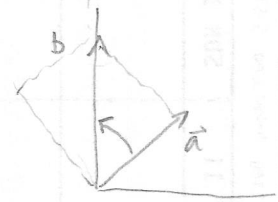
$\emptyset_1 + \emptyset_2 = 2\pi$

$\sin \emptyset_1 = -\sin \emptyset_2$

③ $\text{abs}\left(\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right) =$ Area of ▱ spanned by $\vec{a}$ and $\vec{b}$

Determinants Examples

① $\vec{a} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ $\qquad \vec{b} = [0, 1]$



$$\det \begin{bmatrix} \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \\ 0, & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\|a\| = \sqrt{2\left(\tfrac{1}{\sqrt{2}}\right)^2} = 1 \qquad \|b\| = 1$$

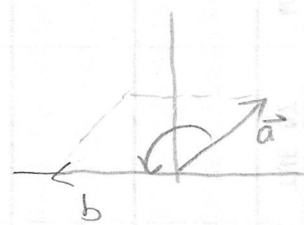
By $\det \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = \|a\| \, \|b\| \sin \emptyset$:

$$\frac{1}{\sqrt{2}} = (1)(1) \sin \emptyset$$

$$\sin \emptyset = \frac{1}{\sqrt{2}}, \qquad \emptyset = \boxed{\pi/4} \text{ or } 3\pi/4$$

② $\vec{a} = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ $\qquad \vec{b} = [-1, 0]$

$$\det \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$



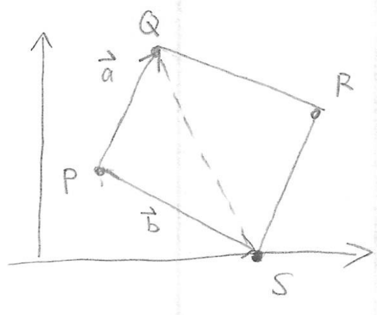
By $\det \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = \|a\| \, \|b\| \sin \emptyset$

$$\frac{1}{\sqrt{2}} = (1)(1) \sin \emptyset$$

$$\sin \emptyset = \frac{1}{\sqrt{2}}, \qquad \emptyset = \pi/4 \text{ or } \boxed{3\pi/4}$$

In both examples, the area of ▱ spanned by both is $\frac{1}{\sqrt{2}}$.

③



$\vec{a} = \vec{PQ} = \vec{Q} - \vec{P} = [1 \;\; 2]$

$\vec{b} = \vec{PS} = \vec{S} - \vec{P} = [3 \;\; -1]$

verify parallel and length PQ = SR

$\vec{SR} = [1 \;\; 2]$

$P = [1 \;\; 1]$

$Q = [2 \;\; 3]$

$R = [5 \;\; 2]$

$S = [4 \;\; 0]$

Finding the area of the ▱ :

$$\text{abs}\left(\det \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}\right) = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = \text{abs}(-1-6) = 7$$

Cross product

determinant of 2 vectors in $\mathbb{R}^3$

$\vec a = [a_1\ a_2\ a_3]$

$\vec b = [b_1\ b_2\ b_3]$

$$\vec a \times \vec b = \det\begin{bmatrix} \hat i & \hat j & \hat k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$\hat i = [1\ 0\ 0]$
$\hat j = [0\ 1\ 0]$
$\hat k = [0\ 0\ 1]$

e.g. $[1\ 1\ 0] \times [0\ 0\ 1] = \begin{vmatrix} \hat i & \hat j & \hat k \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat i \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \hat j \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \hat k \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$

$= \hat i(1) - \hat j(1)$

$= [1,\ -1,\ 0]$

- Properties
  - * $\vec a \times \vec b = -b \times \vec a$
  - $\vec a \times (\vec b \times \vec c) = (\vec c \cdot \vec a)\vec b - (\vec b \cdot \vec a)c$
  - $s(\vec a \times \vec b) = (s\vec a) \times \vec b = \vec a \times (s\vec b)$
  - $\vec a \times (\vec b + \vec c) = \vec a \times \vec b + \vec a \times \vec c$ distributive
  - $\vec a \cdot (\vec b \times \vec c) = (\vec a \times \vec b) \cdot c$

- Geometric Properties

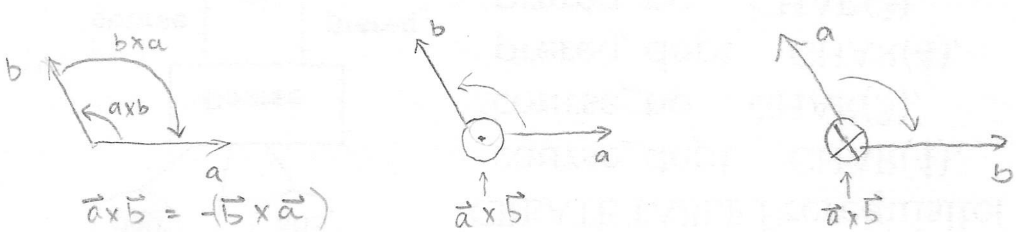
* $\|\vec a \times \vec b\| = \|a\|\ \|b\| \sin\phi, \quad 0 \le \phi \le \pi$

$=$ Area of ▱ spanned by $\vec a$ and $\vec b$ (see determinants)

* $\vec a \times \vec b$ is perpendicular (orthogonal) to both $\vec a$ and $\vec b$

$\vec a, \vec b, \vec a \times \vec b$ satisfy Right Hand Rule if they are non-zero.

Put $\vec a, \vec b$ on same plane. $\vec a \times \vec b$ is $\perp$ to the plane.



$\vec a \times \vec b = -(\vec b \times \vec a)$

⊙ out of page

⊗ into the page

famed arrow head

But usually represented

E.g. Find a unit vector $\perp [1\ 1\ 1]$ and $[-1\ 0\ 0]$

$[1\ 1\ 1] \times [-1\ 0\ 0] = \begin{vmatrix} \hat i & \hat j & \hat k \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = [0, -1, 1]$ The cross product is $\perp$ to both vectors.

$\|[0\ -1\ 1]\| = \sqrt{2}$

Unit Vector: $\frac{1}{\sqrt{2}}[0\ -1\ 1]$

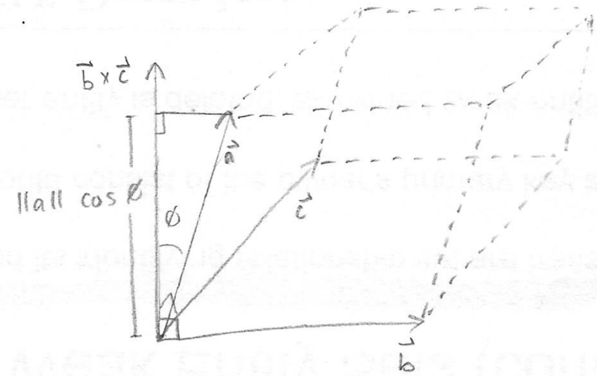
TRIPLE Product

determinant of 3 vectors in $\mathbb{R}^3$

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_3 & c_3 \end{bmatrix} = \vec{a}\cdot(\vec{b}\times\vec{c})$$

* Geometric Meaning:

If $\vec{a}, \vec{b}, \vec{c}$ are non-zero in $\mathbb{R}^3$,



$$\begin{aligned} \vec{a}\cdot(\vec{b}\times\vec{c}) &= \|a\|\,\|\vec{b}\times\vec{c}\|\cos\phi \\ &= (\|a\|\cos\phi)\ \|\vec{b}\times\vec{c}\| \\ &= (\text{height}) \times (\boxed{A} \text{ of } \vec{b}\times\vec{c}) \\ &= \text{volume of the parallelopide spanned by } \vec{a},\vec{b},\vec{c} \quad (\text{signed}) \end{aligned}$$

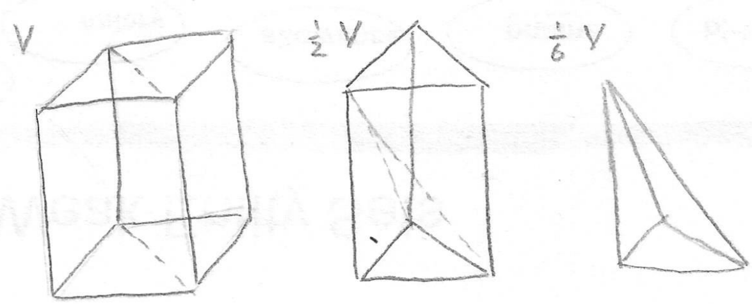
$$\begin{cases} >0 & \text{if } \vec{a},\vec{b},\vec{c} \text{ satisfy Right Hand Rule} \\ <0 & \text{if } \vec{a},\vec{b},\vec{c} \text{ satisfy Left Hand Rule} \end{cases}$$

E.g. $\vec{a} = [1\ 0\ 0]$, $\vec{b} = [0\ 1\ 0]$, $\vec{c} = [0\ 1\ 2]$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$$

$2>0$, so the parallelopiped spanned by $\vec{a},\vec{b},\vec{c}$, satisfies RH Rule, $V = 2$

Volume of tetrahedron with vertices $0, A, B, C$ = $\frac{1}{6} V_{\text{parallelopide}}$



Cross-Product, Determinant Stuff

WW (2.1) Find $\vec{a} \times \vec{b}$. $\vec{a} = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -2 & -4 \\ 1 & -2 & -5 \end{vmatrix} = [10-8, -(20+4), (8+2)]$$

$$= [2, -24, 10]$$

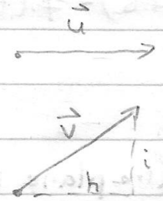
Find $\vec{c} \times \vec{d}$ where $\vec{c} = -4\hat{j} - 5\hat{k}$ and $\vec{d} = 2\hat{i} + 2\hat{j} - \hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -4 & -5 \\ 2 & 2 & -1 \end{vmatrix} = [(4+10), -(10), (8)]$$

$$= [14, -10, 8]$$

WW (2.2) You are looking down at a map. A vector $\vec{u}$ points east with $\|u\| = 5$ and a vector $\vec{v}$ points north-east with $\|v\| = 8$. Describe $\vec{u} \times \vec{v}$.

$\vec{u} = [5, 0]$



$\vec{v} = [\sqrt{32}, \sqrt{32}]$

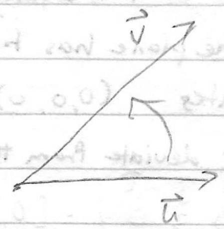
$$\vec{u} \times \vec{v} = \begin{vmatrix} 5 & 0 \\ \sqrt{32} & \sqrt{32} \end{vmatrix} = 5\sqrt{32}$$

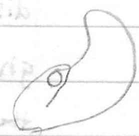
$\sqrt{2i^2} = 8$

$2i^2 = 64$

$i^2 = 32$

$i = \sqrt{32}$



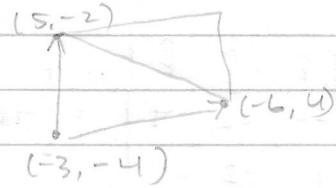


$5\sqrt{32}$ (out of the page)

$[0, 0, 5\sqrt{32}]$

WW. 2.3 Find the area of the triangle with vertices $(-3,-4)$, $(5,-2)$ and $(-6,4)$.

Note: this q can be answered using the determinant of a 2x2 matrix



$(5,-2)-(-3,-4) = [8, 2]$

$(-6,4)-(-3,-4) = [-3, 8]$

$$\begin{vmatrix} 8 & 2 \\ -3 & 8 \end{vmatrix} = 70 = \text{Area of parallelogram},$$

$$\div 2 = 35 = \text{Area of triangle}$$

Find the volume of the parallelopiped with one vertex at $(3,1,-5)$ and adjacent vertices $(9,3,-12)$, $(6,2,-8)$ and $(5,7,-9)$

Note: this q can be answered using the determinant of a 3x3 matrix.

$(9,3,-12)-(3,1,-5) = [6, 2, -7]$

$(6,2,-8)-(3,1,-5) = [3, 1, -3]$

$(5,7,-9)-(3,1,-5) = [2, 6, -4]$

$$\text{abs}\left(\begin{vmatrix} 6 & 2 & -7 \\ 3 & 1 & -3 \\ 2 & 6 & -4 \end{vmatrix}\right) = 6(-4+18) - 2(-12+6) - 7(18-2)$$
$$= 84 + 12 - 112 = |-16| = 16$$

WW. 2.4 Vertical tower built on a horizontal surface (the plane $z=0$). After some time, it is discovered the plane has tilted. Measurements of 3 points on the tower base give coordinates $(0,0,0)$, $(0,3,1)$ and $(1,1,0)$. By what angle does the tower now deviate from the normal?

original $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{vmatrix} = [0, 0, 3]$ (Normal to ground)

slanted $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{vmatrix} = [-1, 1, 3]$ (Normal to the slanted ground)

$$[0,0,3]\cdot[-1,1,3] = (3)(\sqrt{9+1+1})\cos\phi$$
$$\phi = \arccos\left(3\sqrt{11}\right)$$