# Vectors

something with both magnitude and direction.

Mathematically,

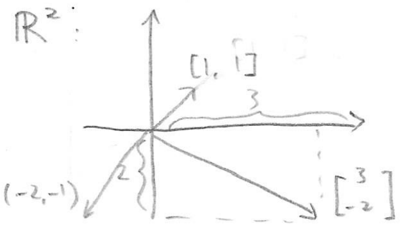
Why bother?

$\mathbb{R}^1$ :


a number

The world is more than 3-dimens

$\mathbb{R}^2$ :

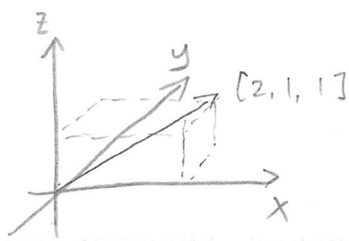



a pair of numbers

$[x_1, x_2]$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

A vector $\vec{x}$ in $\mathbb{R}^{100}$ may be used to represent populations in 100 diff cities.

$\mathbb{R}^3$ :




3 numbers

$[x_1, x_2, x_3]$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

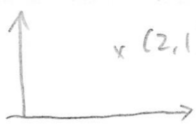
$\mathbb{R}^n$ ?

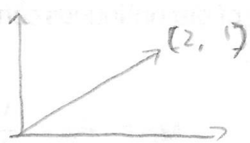
n-tuple of numbers

$\vec{x} = [x_1, x_2, x_3 \ldots x_n]$

or $\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$

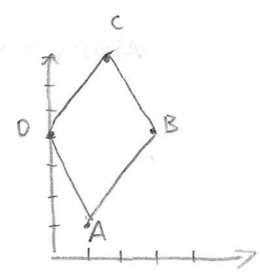
Remark : $X = (x_1, x_2, x_3 \ldots x_n)$ can be regarded as a point in $\mathbb{R}^n$ or as the vector from the origin to the point




Example $A = [1, 1]$, $B = [3, 4]$, $C = [2, 7]$, $D = [0, 4]$

Show that ABCD is a parallelogram



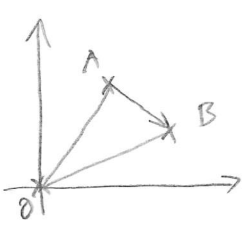

$\vec{AB} = B - A = [3\ 4] - [1\ 1] = [2, 3]$

$\vec{DC} = C - D = [2\ 7] - [0\ 4] = [2, 3]$

ABCD is a parallelogram : $\vec{AB} = \vec{DC}$

$\vec{AB}$ and $\vec{DC}$ have same length and direction

$\vec{AB}$ and $\vec{DC}$ are EQUAL as VECTORS even though they start from different points



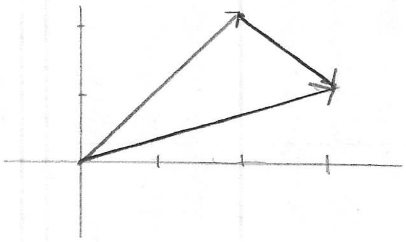

$$\begin{aligned} \vec{AB} &= \vec{AO} + \vec{OB} \\ &= -\vec{OA} + \vec{OB} \\ &= -A + B \\ &= B - A \end{aligned}$$

Examples

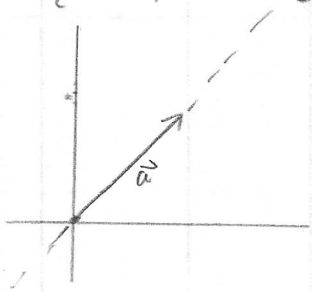
- $2[3, 2] - [0, -1]$
  $= [6, 4] - [0, -1]$
  $= [6, 5]$

- $3[1, -2, 10, 4] + 2[-1, 2, 3, 4]$
  $= [3, -6, 30, 12] + [-2, 4, 6, 8]$
  $= [1, -2, 36, 20]$

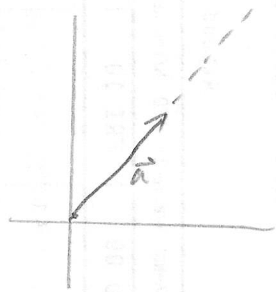
- $[2, 2] + [1, -1] = [3, 1]$



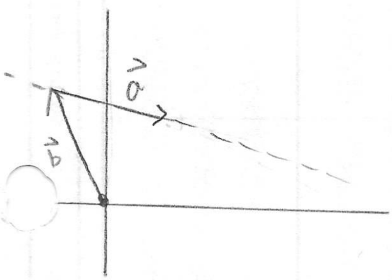
- $\{s\vec{a}\ ;\ s \in \mathbb{R}\}$



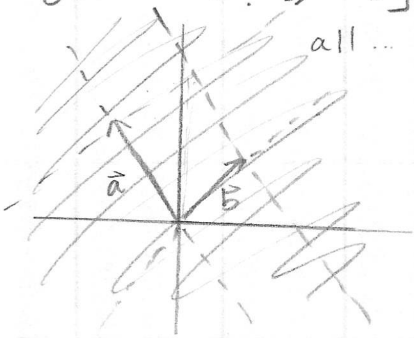

- $\{s\vec{a}\ ;\ s > 0\}$



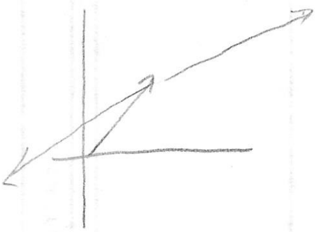

- $\{\vec{b} + s\vec{a},\ s \in \mathbb{R}\}$




- $\{s\vec{a} + t\vec{b} : s, t \in \mathbb{R}\}$



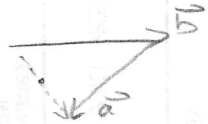

- $\vec{a} + \vec{b} = [a_1 + b_1,\ a_2 + b_2]$ e.g. $\vec{a} = [3\ 4]$ $\quad \vec{a} + \vec{b} = [12, 10]$
  $\vec{a} - \vec{b} = [a_1 - b_1,\ a_2 - b_2]$ $\quad \vec{b} = [9\ 6]$ $\quad \vec{a} - \vec{b} = [-6, -2]$



- midpoint between $\vec{a}$ and $\vec{b}$

$$\frac{\vec{b} - \vec{a}}{2}$$

?

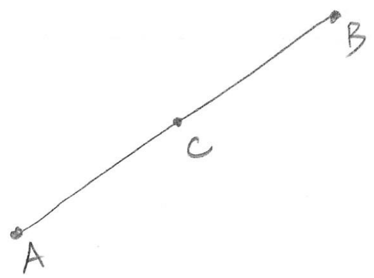




- midpoint between 2 points.

$$\frac{(b + a)}{2}$$

A, B, C are points in $\mathbb{R}^n$

C is the midpoint of AB




$C = (A + B)/2$

$\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AB}$

*coordinate of c :

$$C = \underset{\text{Point}}{A} + \overrightarrow{AC}$$
$$= A + \frac{1}{2}\overrightarrow{AB}$$
$$= A + (\tfrac{1}{2})(B - A)$$
$$= \tfrac{1}{2}A + \tfrac{1}{2}B$$

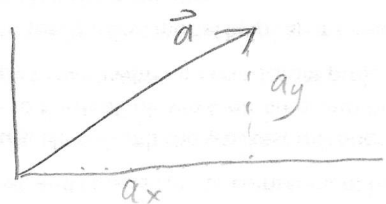
*** Length of a Vector**

$$\boxed{\|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}} = \sqrt{\vec{a}\cdot\vec{a}}$$

applies to any orthogonal coordinate system

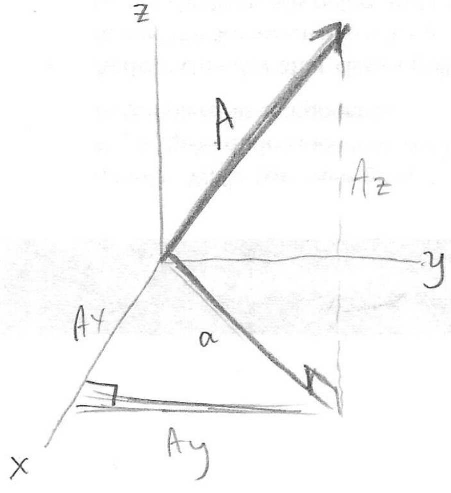
$\|[3 \quad 4 \quad 9 \quad 7]\| = \sqrt{3^2 + 4^2 + 9^2 + 7^2}$

2D :




$\|\vec{a}\| = \sqrt{(a_x)^2 + (a_y)^2}$

3D:

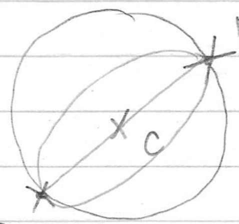



$A = \sqrt{(A_z)^2 + (a)^2}$

$a^2 = (A_x)^2 + (A_y)^2$

$A = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$

Find the equation of the sphere with [1, 5, 2], [3, 3, 4] as the endpoints of its diameters.




$C = A + \frac{1}{2}\overrightarrow{AB}$
$= [1\ 5\ 2] + \frac{1}{2}[2\ -2\ 2]$
$= [2\ 4\ 3]$

$C = \frac{1}{2}(\vec{A} + \vec{B})$
$= [2, 4, 3]$

$r = \|AC\|$
$= \sqrt{(1^2 + 1^2 + 1^2)}$
$= \sqrt{3}$

$\overrightarrow{AC} = [2\ 4\ 3] - [1\ 5\ 2]$
$= [1\ -1\ 1]$

Equation of the sphere:

$\| [x, y, z] - [2, 4, 3] \| = \sqrt{3}$
$\| x-2, y-4, z-3 \| = \sqrt{3}$
$\sqrt{(x-2)^2 + (y-4)^2 + (z-3)^2} = \sqrt{3}$

$(x-2)^2 + (y-4)^2 + (z-3)^2 = 3$

Hilroy

# Dot Product

* $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$ $\quad$ $(\text{length})^2$ = dot product

* $\|\vec{a}\| \, \|\vec{b}\| \cos \varnothing = \vec{a} \cdot \vec{b}$ $\quad$ angle between 2 vectors

In $\mathbb{R}^n$, let $\vec{a} = [a_1 \; a_2 \; a_3 \; \cdots \; a_n]$

$\vec{b} = [b_1 \; b_2 \; b_3 \quad b_n]$

Dot Product: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$

e.g. $[1 \;\; 2] \cdot [3 \;\; 4] = (1)(3) + (2)(4) = 11$

$[1 \;\; 2 \;\; -1] \cdot [3 \;\; 2 \;\; 9] = (1)(3) + (2)(2) + (-1)(9) = -2$

**Properties**

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

  $a_1 b_1 + a_2 b_2 = b_1 a_1 + b_2 a_2$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

  $a_1(b_1 + c_1) = a_1 b_1 + a_1 c_1$
- $s(\vec{a} \cdot \vec{b}) = s\vec{a} \cdot \vec{b} = \vec{a} \cdot s\vec{b}$

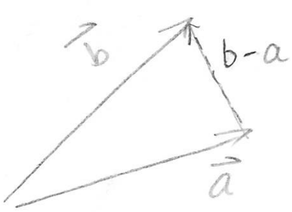
  $s(a_1 b_1) = [sa_1] \cdot [b_1] = a_1 [s b_1]$
- $\vec{0} \cdot \vec{a} = 0$ ← scalar

* $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$[a_1 \;\; a_2] \, [a_1 \;\; a_2] = (a_1)^2 + a_2^2 = \|\vec{a}\|^2$

* $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos \varnothing$

Proof a




$\|b - a\|^2 = (b - a) \cdot (b - a)$

$= \vec{b} \cdot \vec{b} + \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b}$

$= \|b\|^2 + \|a\|^2 - 2\vec{a} \cdot \vec{b}$

cos law

$C^2 = A^2 + B^2 - 2AB \cos \varnothing$

$= \|a\|^2 + \|b\|^2 - 2\|a\| \|b\| \cos$

$\vec{a} \cdot \vec{b} = \|a\| \, \|b\| \cos \varnothing$

Dot Products Examples

- Show that $[1 \ 1 \ 1]$ and $[-1 \ 3 \ 2]$ are $\perp$.

$$[1 \ 1 \ 1] \cdot [-1 \ 3 \ 2] = (1)(-1) + (1)(3) + (1)(2) = 0$$

- Find the angle between $\vec{a} = [1 \ 2 \ -1]$ and $\vec{b} = [-1 \ -1 \ 4]$

$$\vec{a} \cdot \vec{b} = \|a\| \, \|b\| \cos \theta = (1)(-1) + (2)(-1) + (-1)(4) = -7$$

$$= \sqrt{1 + 2^2 + 1} \, \sqrt{1 + 1 + 4^2} \cos \theta = -7$$

$$= \sqrt{6} \, \sqrt{18} \cos \theta = -7$$

$$\theta = \cos^{-1}\left(\frac{-7}{6\sqrt{3}}\right) = 2.31 \text{ rad}$$

Exam: No calculator

- Suppose $\vec{a}, \vec{b}$ are vectors in $\mathbb{R}^n$ with same length.
Show that $\vec{a} + \vec{b} \perp \vec{a} - \vec{b}$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

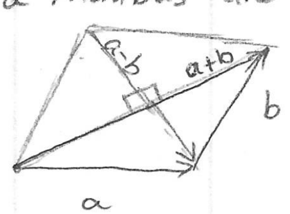
$$= (\vec{a} \cdot \vec{a}) + (\vec{b} \cdot \vec{b}) - (\vec{b} \cdot \vec{b}) - (\vec{b} \cdot \vec{b})$$

$$= (\vec{a} \cdot \vec{a}) - (\vec{b} \cdot \vec{b})$$

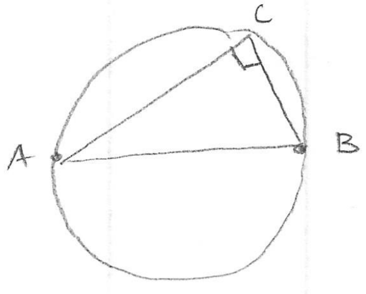
$$= \|\vec{a}\|^2 - \|\vec{b}\|^2 \qquad \|\vec{a}\| = \|\vec{b}\|$$

$$= 0$$

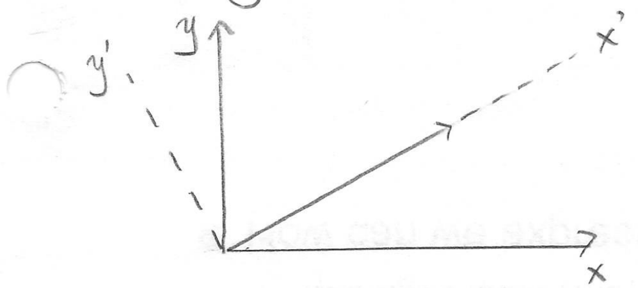
Diagonals in a rhombus are $\perp$. $\vec{a} + \vec{b} \perp \vec{a} - \vec{b}$




- Use dot product to show that $\angle ACB = 90°$



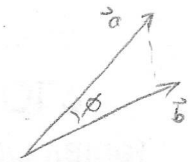
# Projections




[3, 2] on the x'-y' coordinate system is [$\sqrt{13}$ 0]

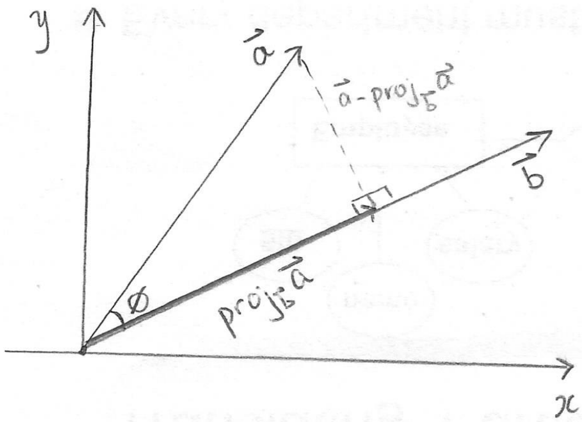
The projection of [3 2] on x'-y' is [$\sqrt{13}$ 0]

$\varnothing < 90°$





$\varnothing > 90°$




$proj_{\vec{b}}\vec{a}$ = the projection of $\vec{a}$ in the direction of $\vec{b}$

the part of $\vec{a}$ that lies in the direction $\vec{b}$

scalar x vector

$$proj_{\vec{b}}\vec{a} = k\vec{b}$$

$$(\vec{a} - proj_{\vec{b}}\vec{a}) \perp \vec{b}$$

Dot Product Property : $\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos \varnothing$

$\perp$ means dot product $= 0$

$$proj_{\vec{b}}\vec{a} = k\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{\|b\|^2}\right)\vec{b}$$

$$(proj_{\vec{b}}\vec{a} - \vec{a}) \cdot \vec{b} = 0$$

$$(k\vec{b} - \vec{a}) \cdot \vec{b} = 0$$

$$k\vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{b} = 0$$

$$k\|b\|^2 = \vec{a} \cdot \vec{b}$$

$$k = \frac{\vec{a} \cdot \vec{b}}{\|b\|^2}$$

$$(\vec{a} - proj_{\vec{b}}\vec{a}) \cdot \vec{b} = 0$$

$$(\vec{a} - k\vec{b}) \cdot \vec{b} = 0$$

$$\vec{a} \cdot \vec{b} - k\vec{b} \cdot \vec{b} = 0$$

$$k\vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{b}$$

$$k = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}$$

Example:

$\vec{a} = [1, 2, 1]$

$\vec{b} = [-1, 1, 1]$

Find $proj_{\vec{b}}\vec{a}$ and $proj_{\vec{a}}\vec{b}$.

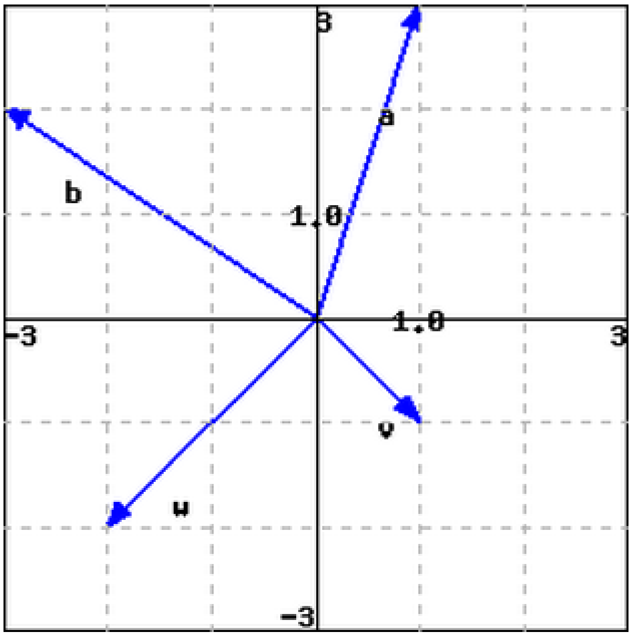
$\vec{a} \cdot \vec{b} = 2$ $\quad \|b\|^2 = \vec{b} \cdot \vec{b} = 3$ $\quad \|a\|^2 = \vec{a} \cdot \vec{a} = 6$

$$proj_{\vec{b}}\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\|b\|^2}\right)\vec{b} = \left(\frac{2}{3}\right)[-1, 1, 1]$$

$$proj_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{\|a\|^2}\right)\vec{a} = \left(\frac{2}{6}\right)[1, 2, 1]$$

## WEBWORK 1: 2.1 to 2.3

(1 pt) Resolve the vectors shown in the figure below into components.




[1,3]
[-3,2]
[1,-1]
[-2,-2]

$\mathbf{a} = [$ ______ , ______ $]$

$\mathbf{b} = [$ ______ , ______ $]$

$\mathbf{v} = [$ ______ , ______ $]$

$\mathbf{w} = [$ ______ , ______ $]$

---

(1 pt) Let $\mathbf{a}$ and $\mathbf{b}$ be the three dimensional vectors and $s$ and $t$ the scalars below

$$\mathbf{a} = \langle 0, 2, 2 \rangle, \quad \mathbf{b} = \langle 5, 6, 8 \rangle, \quad s = 3, \quad t = -2$$

Perform the following operations on these vectors:

[5,8,10]
[0,6,6]
[10,14,18]

(a) $\mathbf{a} + \mathbf{b} =$ ______

(b) $s\mathbf{a} =$ ______

(c) $\mathbf{a} - t\mathbf{b} =$ ______

(1 pt) Find the length of the vectors

**(a)** $\langle 4, 2, 1 \rangle$:

sqrt(21)

Length = ______

**(b)** $\langle -24, -10, -2 \rangle$:

sqrt(24^2+104)

Length = ______

---

(1 pt) Suppose $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$. Then

$\mathbf{u} \cdot \mathbf{v}$= ______ -6

$\mathbf{u} \cdot \mathbf{w}$= ______ 18

$\mathbf{v} \cdot \mathbf{w}$= ______ -24

$\mathbf{v} \cdot \mathbf{v}$= ______ 24

$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$= ______ 12

---

(1 pt) Suppose $\mathbf{u} = \langle -2, 4, 1 \rangle$. Are the following vectors perpendicular to $\mathbf{u}$?

No ▾ **1.** $\langle -2, 3, -3 \rangle$

Yes ▾ **2.** $\langle 3, 8, -26 \rangle$

Yes ▾ **3.** $\langle 2, 1, 0 \rangle$

(1 pt) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{y}$ be the three dimensional vectors

$$\mathbf{a} = \langle 0,4,2 \rangle, \quad \mathbf{b} = \langle 4,5,8 \rangle, \quad \mathbf{c} = \langle 1,5,0 \rangle, \quad \mathbf{y} = \langle -7,3,0 \rangle$$

Perform the following operations on these vectors:

(a) $\mathbf{c} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{y} =$ 20+12 = 32

(b) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{a} =$ [0,36*4,36*2]

(c) $((\mathbf{c} \cdot \mathbf{c})\,\mathbf{a}) \cdot \mathbf{a} =$ [0, 26*4, 26*2]dot[0, 4, 2]
26*16+26*4

---

(1 pt) Find a unit vector in the same direction as $\mathbf{a} = \begin{bmatrix} 1 \\ -5 \\ 9 \end{bmatrix}$

1/(sqrt(81+25+1)) * [1, -5, 9]

---

Let $\mathbf{a} = \langle 2,-7,-7 \rangle$ and $\mathbf{b} = \langle 10,6,9 \rangle$ be vectors. Find the vector projection of $\mathbf{b}$ in the direction of $\mathbf{a}$.

[2, -7, -7] ( [10,6,9]dot[2,-7,-7] / 4+2*49 )

---

The figure shows a rectangular box in three-dimensional space that contains several vectors. (The vector $\mathbf{c}$ is in the $xz$-plane, and the vector $\mathbf{e}$ is in the $xy$-plane.)

Are the following statements true or false?

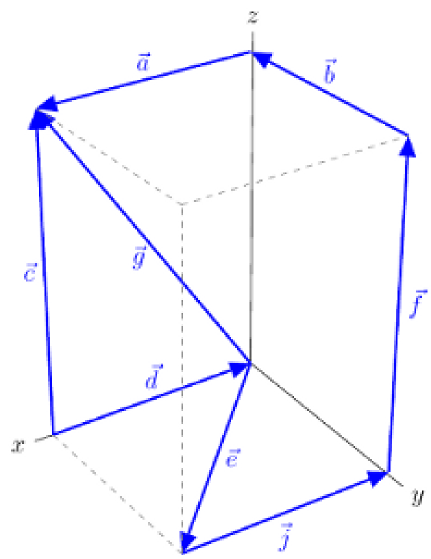
True 1. $\mathbf{e} + \mathbf{b} = \mathbf{g} - \mathbf{f}$

False 2. $\mathbf{a} = \mathbf{d}$

False 3. $\mathbf{g} = \mathbf{j} + \mathbf{f}$

True 4. $\mathbf{e} = \mathbf{a} - \mathbf{b}$

False 5. $\mathbf{d} = \mathbf{g} - \mathbf{c}$

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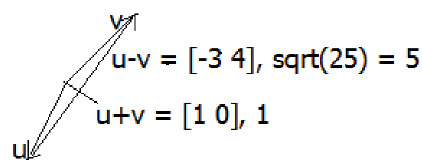
(1 pt) Suppose $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ are two vectors that form the sides of a parallelogram. Then the lengths of the two diagonals of the parallelogram are

*Separate your answers with a comma.*

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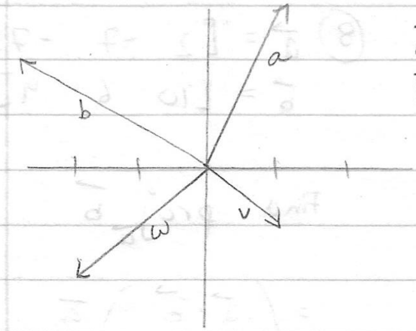
# Webwork 1

(1)




$\vec{a} = [1 \ \ 2]$

$\vec{b} = [-3 \ \ 2]$

$\vec{v} = [1 \ \ -1]$

$\vec{w} = [-2 \ \ -2]$

(2) $\vec{a} = [0 \ \ 2 \ \ 2]$ $\quad s = 3$

$\vec{b} = [5 \ \ 6 \ \ 8]$ $\quad t = -2$

(2) a) $\vec{a} + \vec{b}$

$= [0 \ \ 2 \ \ 2] + [5 \ \ 6 \ \ 8]$

$= [5 \ \ 8 \ \ 10]$

b) $s\vec{a}$

$= 3\,[0 \ \ 2 \ \ 2]$

$= [0 \ \ 6 \ \ 6]$

c) $\vec{a} - t\vec{b}$

$= [0 \ \ 2 \ \ 2] - (-2)[5 \ \ 6 \ \ 8]$

$= [10 \ \ 14 \ \ 18]$

(3) Find the length of the vectors

a) $[4 \ \ 2 \ \ 1]$

$\|[4 \ \ 2 \ \ 1]\| = \sqrt{4^2+2^2+1}$

$= \sqrt{21}$

b) $[-24 \ \ -10 \ \ -2]$

$\|[-24 \ \ -10 \ \ -2]\| = \sqrt{24^2+10^2+2^2}$

(4) $\vec{u} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$

$\vec{u} \cdot \vec{v} = (-2)(-2) + (4)(-4) + (3)(2) = -6$

$\vec{u} \cdot \vec{w} = -8 + 20 + 6 = 18$

$\vec{v} \cdot \vec{w} = -8 - 20 + 4 = -24$

$\vec{v} \cdot \vec{v} = 4 + 16 + 4 = 24$

$\vec{u} \cdot (\vec{v} + \vec{w}) = (-2)(2) + 4(1) + 3(4) = 12$

(5) $\vec{u} = [-2 \ \ 4 \ \ 1]$

Are the following perpendicular?

a) $[-2 \ \ 3 \ \ -3]$

$[-2 \ 4 \ 1] \cdot [-2 \ 3 \ -3] = 4 + 12 - 3 = 13$ NO

b) $[3 \ \ 8 \ \ -26]$

$[-2 \ 4 \ 1] \cdot [3 \ 8 \ -26] = -6 + 32 - 26 = 0$ Yes

c) $[2 \ \ 1 \ \ 0]$

$[-2 \ 4 \ 1] \cdot [2 \ 1 \ 0] = -4 + 4 = 0$ Yes

(6) $\vec{a} = [0\ 4\ 2]$

$\vec{b} = [4\ 5\ 8]$

$\vec{c} = [1\ 5\ 0]$

$\vec{y} = [-7\ 3\ 0]$

a) $\vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y}$

$= [1\ 5\ 0] \cdot [0\ 4\ 2] + [0\ 4\ 2][-7\ 3\ 0]$

$= 20 + 12 = 32$

b) $(\vec{a} \cdot \vec{b})\vec{a}$

$= ([0\ 4\ 2] \cdot [4\ 5\ 8])[0\ 4\ 2]$

$= (20 + 16)\ [0\ 4\ 2]$

$= [0\ 144\ 72]$

c) $((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a}$

$= ((1^2 + 5^2)[0\ 4\ 2]) \cdot [0\ 4\ 2]$

$= [0\ 104\ 52] \cdot [0\ 4\ 2]$

$= 416 + 104$

$= 520$

(8) $\vec{a} = [2\ -7\ -7]$

$\vec{b} = [10\ 6\ 9]$

Find $\text{proj}_{\vec{a}} \vec{b}$

$= \left( \frac{\vec{a} \cdot \vec{b}}{\|a\|^2} \right) \vec{a}$

$= \left( \frac{20 - 42 - 63}{2^2 + 7^2 + 7^2} \right) [2\ -7\ -7]$

$= \left( \frac{-85}{102} \right) [2\ -7\ -7]$

(7) Find a unit vector in the same direction as $\vec{a} = \begin{bmatrix} 1 \\ -5 \\ 9 \end{bmatrix}$

$\|\vec{a}\| = \sqrt{1^2 + 5^2 + 9^2} = \sqrt{107}$

$\vec{U}_a = \begin{bmatrix} \frac{1}{\sqrt{107}} \\ \frac{-5}{\sqrt{107}} \\ \frac{9}{\sqrt{107}} \end{bmatrix}$

$\sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

Unit vector has length of 1

$U_x = 1 = \frac{a_x}{\|\vec{a}\|}$

(9) see diagram

Math 101 webwork

due Wed at 9pm

$\vec{e} + \vec{b} = \vec{g} - \vec{f}$ TRUE

$\vec{a} = \vec{d}$ FALSE

$\vec{g} = \vec{j} + \vec{f}$ FALSE

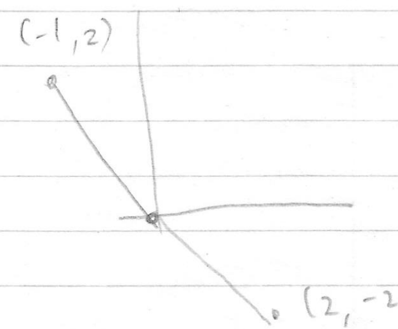
$\vec{e} = \vec{a} - \vec{b}$ TRUE

$\vec{d} = \vec{g} - \vec{c}$ FALSE

(10) $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

2 vectors, sides of parallelogram.

Lengths of the 2 DIAGONALS.




diagonals = $\vec{u} - \vec{v}$ $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $\sqrt{25}$

$\vec{v} - \vec{u} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ $\sqrt{25}$

Diagonals:

$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 1

$\vec{u} - \vec{v}$

and $\vec{u} + \vec{v}$