Problem No. 1 Give the general solutions of the following differential equations

(a) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

(b) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

SOLUTION

a

$$r^2 - 4r + 3 = 0 \rightarrow r_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} = 2 \pm 1 \rightarrow r_1 = 3 \text{ and } r_1 = 1$$

$$y = A\,e^{3x} + B\,e^{x}$$

b

$$r^2 + 2r + 2 = 0 \rightarrow r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{4 \pm 2\,i}{2} = -1 \pm 1\,i$$

$$y = e^{-x}\left\{A \cos x + B \sin x\right\}$$

Problem no 2. Provide the general solution of the following differential equation by undetermined coefficients

$$y'' + 4y' + 4y = x^2 - 2x$$

SOLUTION

$$r^2 + 4r + 4 = 0 \rightarrow r_{1,2} = \frac{-4 \pm \sqrt{16-16}}{2} = -\frac{4}{2} = -2$$

$$y_h = \left\{A + B\,x\right\}e^{-2x}$$

Assume

$$y_p = a\,x^2 + b\,x + c \rightarrow y_p{}' = 2a\,x + b \rightarrow y_p{}'' = 2a$$

$$2a + 4\left(2a\,x + b\right) + 4\left(a\,x^2 + b\,x + c\right) = x^2 - 2\,x + 0$$

$$4ax^2 + \left(8a + 4b\right)x + \left(2a + 4c + 4b\right) = x^2 - x + 0$$

$$y_p = \frac{1}{4}x^2 - x + \frac{7}{8}$$

$$a = 1/4,\ b = -1,\ c = 7/8$$

$$y = \{A + Bx\}e^{-2x} + \frac{1}{4}x^2 - x + \frac{7}{8}$$

Problem no 3 Give the general solution of the following differential equation using the variation of parameters

$$y'' + y = \sec^3 x$$

SOLUTION

$$r^2 + 1 = 0$$

$$r = \pm i = 0 \pm i.$$

$$y_h = e^0 (C_1 \cos x + C_2 \sin x)$$

LET

$$y_1 = \sin x \quad \text{AND} \quad y_2 = \cos x$$

$$y = (\sin x)v_1 + (\cos x)v_2 .$$

$$(\sin x)v_1' + (\cos x)v_2' = 0$$

$$(\sin x)' v_1' + (\cos x)' v_2' = \sec^3 x .$$

$$\boxed{\begin{aligned}(\sin x)v_1' + (\cos x)v_2' &= 0 \\ (\cos x)v_1' - (\sin x)v_2' &= \sec^3 x\end{aligned}}$$

2.

$$(\sin x) v_1' + (\cos x) v_2' = 0 \quad \Big| \sin x$$

$$(\cos x) v_1' - (\sin x) v_2' = \sec^3 x \quad \Big| \cos x$$

$$(\sin^2 x) v_1' + (\cos x \sin x) v_2' = 0$$

$$(\cos^2 x) v_1' - (\sin x \cos x) v_2' = \sec^3 x \cos x$$

$$(\underbrace{\sin^2 x + \cos^2 x}_{1}) v_1' = \sec^2 x$$

$$\frac{dv_1}{dx} = v_1' = \sec^2 x$$

$$v_1 = \int \sec^2 x \, dx + C_1$$

$$\boxed{v_1 = \tan x + C_1}$$

3A.

$$v_1' = \sec^2 x$$

$$v_2' = -\frac{\sin x}{\cos x} v_1' = -\frac{\sin x}{\cos x} \sec^2 x$$

$$v_2' = -\frac{\sin x}{\cos^3 x} = \frac{dv_2}{dx}$$

$$v_2 = -\int \frac{\sin x}{\cos^3 x} dx + C_2 .$$

$$= -\frac{1}{2} \sec^2 x + C_2$$

$$y = y_1 v_1 + y_2 v_2$$

$$= \sin x \{ \underbrace{\tan x + C_1}_{v_1} \}$$

$$+ \cos x \left\{ -\frac{1}{2} \sec^2 x + C_2 \right\}$$

$$y = C_1 \sin x + C_2 \cos x + \frac{\sin^2 x}{\cos x} - \frac{1}{2\cos x}$$

$$\begin{Bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{Bmatrix} \begin{Bmatrix} v_1' \\ v_2' \end{Bmatrix} = \begin{Bmatrix} 0 \\ \sec^3 x \end{Bmatrix}$$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\left(\sin^2 x + \cos^2 x\right) = -1$$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sec^3 x & -\sin x \end{vmatrix}}{W} = -\left(-\sec^3 x \cos x\right) = \sec^2 x$$

$$\frac{dv_1}{dx} = \sec^2 x \rightarrow v_1 = \int \sec^2 x \, dx = \tan x + C_1$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec^3 x \end{vmatrix}}{W} = -\sin x \sec^3 x$$

$$v_2 = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec^3 x \end{vmatrix}}{W} = -\int \sin x \sec^3 x \, dx = -\frac{1}{2}\sec^2 x + C_2$$

$$y = \sin x \left(\tan x + C_1\right) + \cos\left(-\frac{1}{2}\sec^2 x + C_2\right) = C_1 \sin x + C_2 \cos x + \frac{\sin^2 x}{\cos x} - \frac{1}{2\cos x}$$

Problem No. 4 Solve the given boundary value problem

$$y'' + 4y' + 5y = 35\, e^{-4x}, \quad y(0) = -3, \; y'(0) = 1$$

SOLUTION

$$r^2 + 4r + 5 = 0 \rightarrow r_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = -2 \pm 1$$

$$y_h = e^{-2x}\left\{A \cos x + B \sin x\right\}$$

ASSUME

$y_p = c\, e^{-4x} \rightarrow y_p' = -4\, c\, e^{-4x} \rightarrow y_p'' = 16\, c\, e^{-4x}$

$16\, c\, e^{-4x} + 4\left(-4\, c\, e^{-4x}\right) + 5\left(c\, e^{-4x}\right) = 35\, e^{-4x}$

$c = 7$

or

$y = e^{-2x}\left\{A \cos x + B \sin x\right\} + 7\, e^{-4x} \quad \rightarrow$

$y' = -2e^{-2x}\left\{A \cos x + B \sin x\right\} + e^{-2x}\left\{A \sin x + B \cos x\right\} - 28\, e^{-4x}$

$y(0) = e^{0}\left\{A \cos 0 + B \sin 0\right\} + 7\, e^{0} = -3 \rightarrow A = -3 - 7 = -10$

$y'(0) = -2\left\{A\right\} + \left\{B\right\} - 28 = 1 \rightarrow 20 + B - 28 = 1 \rightarrow B = 9$

$y = e^{-2x}\left\{-10 \cos x + 9 \sin x\right\} + 7\, e^{-4x}$