BOOK No. ____________ **CONCORDIA UNIVERSITY**

NUMBER OF BOOKS USED ______ ADDITIONAL BOOK (to be placed inside back cover of first book)

FILL IN THE FOLLOWING:

NAME Solution midterm 2 Fall 2015
(Please Print) SURNAME GIVEN NAMES

SUBJECT ____________
(Course and Number)

COURSE GIVEN BY Chadi Assi

EXAMINATION SUPERVISED ____________

DATE WRITTEN ____________

1) $S = \{ n \in \mathbb{N} \mid 0 \leq n \leq 11 \}$

$f: S \to S$: $f(n)$ remainder when $5n+2$ is divided by 12

$n=0$, $f(0) = ?$ $5\times 0 + 2 = 2$; when 2 is divided by 12, we get a remainder of 2

$\Rightarrow \boxed{f(0) = 2}$

$n=1$: $5\times 1 + 2 = 7$, $\boxed{f(1) = 7}$

$n=2$: $5\times 2 + 2 = 12$ $\boxed{f(2) = 0}$

$n=3$: $5\times 3 + 2 = 17$ $\boxed{f(3) = 5}$ (5 is remainder of $\frac{17}{12}$)

$n=4$: $5\times 4 + 2 = 22$ $\boxed{f(4) = 10}$ ($\frac{22}{12} = 1(10)$ remainder

$n=5$: $5\times 5 + 2 = 27 = 2(12) + 3$

$\Rightarrow \boxed{f(5) = 3}$

$n=6$: $5\times 6 + 2 = 32 = 2(12) + 8 \Rightarrow \boxed{f(6) = 8}$

$n=7$: $5\times 7 + 2 = 37 = 3(12) + 1 \Rightarrow \boxed{f(7) = 1}$

$n=8$: $5\times 8 + 2 = 42 = 3(12) + 6 \Rightarrow \boxed{f(8) = 6}$

$n=9$: $5\times 9 + 2 = 47 = 3(12) + 11 \Rightarrow \boxed{f(9) = 11}$

$n=10$: $5\times 10 + 2 = 52 = 4(12) + 4 \Rightarrow \boxed{f(10) = 4}$

$n=11$: $5\times 11 + 2 = 57 = 4(12) + 9 \Rightarrow \boxed{f(11) = 9}$

$\Rightarrow$ $\boxed{f \text{ is one-to-one}}$, $\boxed{f \text{ is onto}}$

2)

$f(x) = ax + b$

$g(x) = cx + d$

$f \circ g(x) = f(g(x)) = f(cx+d) = a(cx+d) + b$

$= acx + ad + b$

$g \circ f(x) = g(ax+b) = c(ax+b) + d = cax + cb + d$

$f \circ g = g \circ f$

$\Rightarrow a\cancel{c}x + ad + b = \cancel{ca}x + cb + d$

$\Rightarrow \boxed{ad + b = cb + d}$

3) $\sum_{k=3}^{22} (3k+4)^2 = \sum_{k=3}^{22} \left((3k)^2 + 24k + 16\right)$

$= \sum_{k=3}^{22} 9k^2 + \sum_{k=3}^{22} 24k + \sum_{k=3}^{22} 16$

$= 9\sum_{k=3}^{22} k^2 + 24\sum_{k=3}^{22} k + \sum_{k=3}^{22} 16$

$= 9\left[\sum_{k=0}^{22} k^2 - \sum_{k=0}^{2} k^2\right] + 24\left[\sum_{k=0}^{22} k - \sum_{k=0}^{2} k\right] + \left[(22-3+1)\,16\right]$

$= 9\left[\frac{22(23)(45)}{6} - \frac{2 \times 3 \times 5}{6}\right] + 24\left[\frac{22 \times 23}{2} - \frac{2 \times 3}{2}\right] + 16 \times 20$

$= 9\left[3795 - 5\right] + 24\left[253 - 3\right] + 320 = 40421$

4) $x \in \mathbb{R}$ $\quad n \in \mathbb{Z}$

$x \le n$ iff $\lceil x \rceil \le n$ $\iff$ $x \le n \longleftrightarrow \lceil x \rceil \le n$

① $x \le n \longrightarrow \lceil x \rceil \le n$

proof by contraposition: $\lceil x \rceil > n \longrightarrow x > n$

~~suppose~~

$\lceil x \rceil > n \Rightarrow$ we can write $\lceil x \rceil = n+1$ (second highest integer to n

$\Rightarrow \quad x = (n+1) - \varepsilon \qquad 0 \le \varepsilon < 1$

$\quad = n + (1-\varepsilon) \qquad$ where $1-\varepsilon \neq 0$

$1 - \varepsilon > 0$

$\Rightarrow \quad x > n$

$\Rightarrow$ ① is shown.

② $\lceil x \rceil \le n \Longrightarrow x \le n$

~~let $\lceil x \rceil = m \Rightarrow x = m - \varepsilon$ $\quad 0 \le \varepsilon < 1$~~

~~since $\lceil x \rceil \le n \Rightarrow m \le n$~~

contraposition:

$x > n \longrightarrow \lceil x \rceil > n$?

$x > n \Rightarrow$ let $x = n + \varepsilon \qquad 1 \ge \varepsilon > 0$

$= \cancel{(n+1) + 1 - \varepsilon}$

$= (n+1) + (\varepsilon - 1)$

$= (n+1) - (1-\varepsilon) \qquad 0 \le 1-\varepsilon < 1$

$\Rightarrow \quad \lceil x \rceil = n+1 > n$

$\Rightarrow \lceil x \rceil \le n \longrightarrow x \le n$

5) $\exists N$ s.t. $3^n < n!$ $n > N$

$P(n): 3^n < n!$

$n=0: 3^0 < 0!$ ✗

$n=1: 3^1 < 1!$ ✗

$n=2: 3^2 < 2!$ ✗

$n=3: 3^3 < 3!$ ✗

$n=4: 3^4 < 4!$ ✗

$n=5: 3^5 < 5!$ ✗

$n=6: 3^6 < 6!$ ✗

$n=7: 3^7 < 7!$ ✓

let $P(n): 3^n < n!$ $n \geq 7$ $\Rightarrow N = 7$

<u>basis step</u>: $P(7)$ is true

<u>inductive step</u> $P(k) \longrightarrow P(k+1)$

$$3^k < k! \implies 3^{(k+1)} < (k+1)!$$

let $3^k < k!$

$\Rightarrow 3 \times 3^k = 3^{k+1} < 3 \times k!$

$< (k+1)\, k!$ sine $3 < k+1$ for $k \geq 7$

$\Rightarrow 3^{k+1} < (k+1)!$

by induction $P(n)$ is T.

6) $P(n)$: 21 divides $4^{n+1} + 5^{2n-1}$ $\quad n > 0$

$P(1)$: 21 divides $4^{1+1} + 5^{2(1)-1} = 4^2 + 5 = 21$

true since 21 divides 21.

$P(k)$: 21 divides $4^{k+1} + 5^{2k-1}$

$P(k+1)$: 21 divides $4^{k+1+1} + 5^{2(k+1)-1}$

$P(k) \longrightarrow P(k+1)$?

$$4^{k+1+1} + 5^{2(k+1)-1} = 4.4^{k+1} + 5^2\, 5^{2k-1}$$

$$= 4\, 4^{k+1} + (4+21)\, 5^{2k-1}$$

$$= 4\, 4^{k+1} + 4.5^{2k-1} + 21.5^{2k-1}$$

$$= 4\underbrace{\left(4^{k+1} + 5^{2k-1}\right)}_{\text{using } P(k) \text{ divisible by 2}} + \underbrace{21.5^{2k-1}}_{\text{divisible by 2}}$$

$\Rightarrow$ $P(k+1)$ is T

$\Rightarrow$ by induction, $P(n)$ is true

7) $a_n + 4a_{n-1} = 0 \qquad a_1 = -12$

$\Rightarrow a_n = (-4)\, a_{n-1}$

a)

$a_2 = -4 a_1 = -4(-12) = 48$

$a_3 = -4 a_2 = -4(48)$

$a_4 = -4 a_3 = -4(-4)(48) = 16 \times 48$

b) $a_n = -4a_{n-1}$

$a_{n-1} = -4a_{n-2}$

$a_{n-2} = -4a_{n-3} \quad \Rightarrow a_{n-2} = (-1)^{n-3} 4^{n-3} a_1$

$\vdots$

$a_4 = -4a_3 \quad \Rightarrow a_4 = -4(-4)(-4)a_1 = (-1)^3 4^3 a_1$

$a_3 = -4a_2 \Rightarrow a_3 = (-4)(-4)a_1 = (-1)^2 4^2 a_1$

$a_2 = -4a_1$

$$\Rightarrow \boxed{a_n = (-1)^{n-1} 4^{n-1} a_1}$$

<u>verify</u>:

$$a_n \overset{?}{=} -4a_{n-1}$$

$$(-1)^{n-1} 4^{n-1} \cancel{a_1} \overset{?}{=} -4 \cdot (-1)^{n-2} 4^{n-2} \cancel{a_1}$$

$$(-1)^{n-1} 4 \cdot \cancel{4^{n-2}} = -4 \cdot (-1)^{n-2} \cancel{4^{n-2}}$$

$$\cancel{(-1)^{n-2}} \cancel{(-1)4} = \cancel{-4} \cancel{(-1)^{n-2}}$$

8) $A = \{2,3,4,5\}$

$B = \{2,4,6\}$

a)

$A \times B$: $R_1 : \{(m,n) \mid m-n = 1\} = \{(3,2),(5,4)\}$

$R_2 = \{(m,n) \mid m \mid n\} = \{(2,2),(2,4),(2,6),(3,6),(4,4)\}$

b) $R_1 = \{(3,2), (5,4)\}$

R_1 is not reflexive
R_1 is not symmetric
R_1 is antisymmetric
R_1 is transitive

$R_2 = \{(2,2), (2,4), (2,6), (3,6), (4,4)\}$

R_2 is not reflexive
R_2 is not symmetric
R_2 is antisymmetric
R_2 is transitive

9) $S = \mathbb{Z} \times \mathbb{Z}^+$
(p,q) and $(r,s) \in S$ and equivalent iff: $ps = qr$

a) $\rightarrow$ S is reflexive
$(p,q)\ S\ (p,q)$ since $p \times q = q \times p$

$\rightarrow$ S is symmetric :
$(p,q)\ S\ (r,s) \longrightarrow (r,s)\ S\ (p,q)$?

$ps = qr \Rightarrow qr = ps \Rightarrow rq = sp \Rightarrow (r,s)\ S\ (p,q)$

$\rightarrow$ S is transitive
$(p,q)\ S\ (r,s) \wedge (r,s)\ S\ (u,v) \rightarrow (p,q)\ S\ (u,v)$?
$ps = qr \wedge \underbrace{rv = su}$

$\frac{p}{q} = \frac{r}{s} \longrightarrow \frac{r}{s} = \frac{u}{v} \Rightarrow \frac{p}{q} = \frac{u}{v} \Rightarrow pv = qu \Rightarrow$
$\Rightarrow S$ is transitive
$\Rightarrow S$ is equivalence

b) S is equivalence,
equivalence class of $(3,8)$:

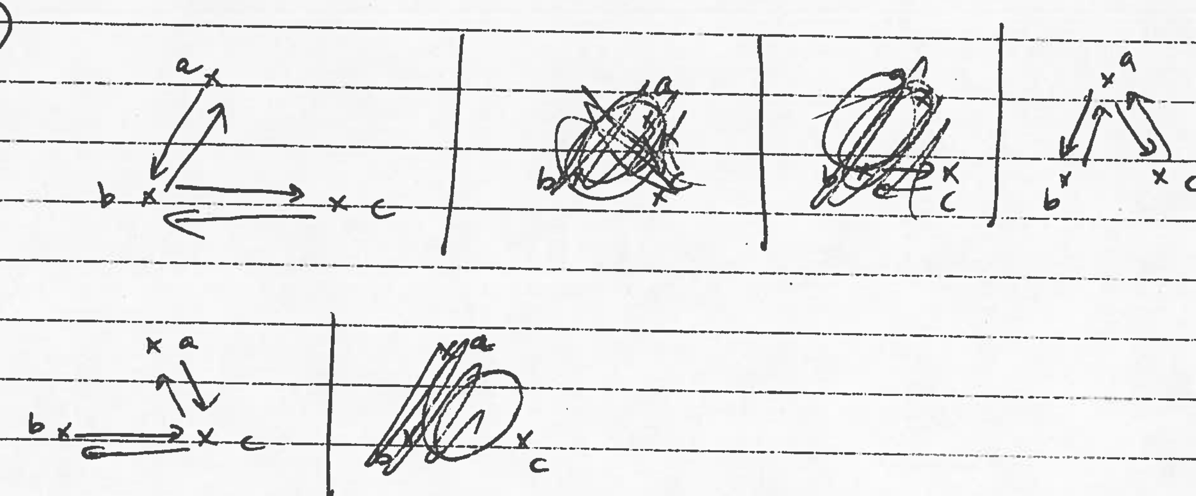
$$(3,8)\, S\, (r,s) \Rightarrow 3s = 8r$$

$$\Rightarrow \frac{s}{r} = \frac{8}{3}$$

equivalence class: $\binom{8}{3} \times k$, $k = 2, 3, 4, \ldots$

10) $\{a, b, c\}$

a)



b)