Question 1:

In the computer market, the price of a computer normally depends on the speed that it can run. High speed computers normally have higher price. Assume that in the market 10% of computers have high price, 30% have medium price and 60% have low price. In the past, 90% of high price computers, 50% of medium price computers and 10% of low price computers have had low speed.

a) What is the probability that a randomly selected computer does not have low speed?

b) If a computer is low speed what is the probability that the computer is not sold at low price?

Solution 1:

HP: High Price *MP*: Medium Price *LP*: Low Price

LS: Low Speed

Based on the question, we have following probabilities:

$P(HP) = 0.10, \qquad P(MP) = 0.30, \qquad P(LP) = 0.60$

$P(LowSpeed \mid HP) = 0.9, \qquad P(LowSpeed \mid MP) = 0.5, \qquad P(LowSpeed \mid LP) = 0.1$

a) (2.5 marks)

Based on total probability rule:

$$P(LowSpeed) = P(LowSpeed \mid HP)P(HP) + P(LowSpeed \mid MP)P(MP) + P(LowSpeed \mid LP)P(LP)$$
$$P(LowSpeed) = 0.9 \times 0.10 + 0.5 \times 0.30 + 0.1 \times 0.60$$
$$P(LowSpeed) = 0.30$$

Probability that a randomly selected computer does not have low speed $= 1 - 0.30 = 0.70$

b) (2.5 marks)

$$P(LP \mid LowSpeed) = \frac{P(LowSpeed \mid LP)P(LP)}{P(LowSpeed)} = \frac{0.1 \times 0.6}{0.30} = \frac{0.06}{0.30}$$
$$P(\overline{LP} \mid LowSpeed) = 1 - P(LP \mid LowSpeed) = 1 - \frac{0.06}{0.30} = 0.8$$

Probability that the a computer does not have low price, given that the computer is low speed is

$P(\overline{LP} \mid LowSpeed) = 0.8$

Question 2:

The receptionist of a company receives phone calls regularly. The time between phone calls in a typical day is known to be exponentially distributed with mean of 2 minutes.

a) What is the probability that the time between two received calls is between 2 and 5 minutes?

b) What is the probability that the receptionist receives four to six calls between 10:30AM up to 10:45 AM on a specific day?

Solution 2:

Part a)

X : Random variable denoting time between two phone calls

$$\mu = 2, \quad \lambda = \frac{1}{\mu} = \frac{1}{2} \quad \text{and} \quad f_X(x) = \frac{1}{2} e^{-\frac{x}{2}}$$

$$P(2 < X < 5) = \int_2^5 f_X(x)dx = \int_2^5 \frac{1}{2} e^{-\frac{x}{2}} dx = -e^{-\frac{x}{2}}\Bigg|_2^5 = \left(-e^{-\frac{5}{2}} + e^{-\frac{2}{2}}\right)$$

Answer: $P(2 < X < 5) = 0.286$

Part b)

Let Y denote the number of calls in a specific interval, then Y is Poisson random variable. Because of memoryless property of Poisson distribution the interval is 10:45-10:30=15 minutes and therefore,

$\lambda = \frac{15}{2} = 7.5$ and $f_Y(y) = \frac{e^{-\lambda}\lambda^y}{y!} = \frac{e^{-7.5}7.5^y}{y!}$.

$$P(4 \le Y \le 6) = P(Y = 4) + P(Y = 5) + P(Y = 6) = f_Y(4) + f_Y(5) + f_Y(6) = \frac{e^{-7.5}7.5^4}{4!} + \frac{e^{-7.5}7.5^5}{5!} + \frac{e^{-7.5}7.5^6}{6!}$$

Answer: $P(4 \le Y \le 6) = 0.319$

Question 3:

The function $f_{XY}(x,y) = cxy^2$ is a joint probability density function over the range of $0 \le x \le 2,\ 0 \le y \le 5$.

a) Determine the constant "c". Are random variables X and Y independent? Justify your answer.

b) Determine the conditional density function of Y given $X = 1.5$. Calculate $E(Y)$ and $V(Y)$.

Solution 3:

3a)

$$\int_{x=0}^{2}\int_{y=0}^{5} cxy^2 dxdy = 1 \rightarrow \int_{0}^{2} \frac{cxy^3}{3}]_0^5\, dx = \int_0^2 \frac{cx}{3} \times 125 = \frac{125}{3} c \left[\frac{x^2}{2}\right]_0^2 = \frac{250c}{3} = 1 \rightarrow c = \frac{3}{250}$$
$$= 0.012$$

$$f_x(x) = \int_0^5 cxy^2 dy = \frac{c}{3} xy^3]_0^5 = \frac{125xc}{3} = \frac{x}{2}$$
$$f_y(y) = \int_0^2 cxy^2 dx = \frac{cy^2}{2} x^2]_0^2 = 2cy^2$$

$$\Longrightarrow f_x(x) f_y(y) = cxy^2 \longrightarrow f_{xy}(x,y) = f_x(x) f_y(y) \longrightarrow So\ x, y\ are\ independent$$

3b)

$$f(y)_{y|x=1.5} = \frac{f(1.5,y)}{f(1.5)} = \frac{\frac{3}{250} \times 1.5 \times y^2}{\frac{125}{3} \times \frac{3}{250} \times 1.5} = \frac{cxy^2}{\frac{125}{3} xc} = \frac{3}{125} y^2 = 0.024y^2$$

$$E(y) = \int_{x=0}^{2}\int_{y=0}^{5} y(cxy^2) dxdy =$$
$$\rightarrow E(y) = c \int_{x=0}^{2}\int_{y=0}^{5} xy^3 dxdy = c\int_0^2 \frac{x}{4}[y^4]_0^5\, dx = c\int_0^2 \frac{625}{4} xdx = \frac{625c}{8} x^2]_0^2$$
$$E(y) = \frac{625c}{8} \times 4 = \frac{625}{2} \times \frac{3}{250} = \frac{15}{4} = 3.75\ \rightarrow E(y) = \frac{15}{4} = 3.75$$
$$V(y) = \int_{x=0}^{2}\int_{y=0}^{5} y^2(cxy^2) dxdy - \mu^2 = c\int_0^2 x\frac{y^5}{5}]_0^5 dx - \mu^2 = c\int_0^2 625xdx - \mu^2 =$$
$$= \frac{625c}{2} x^2]_0^2 - \mu^2 = 625 \times 2c - \mu^2 = \frac{125c \times 3}{25c} - \left(\frac{15}{4}\right)^2 = 15 - \left(\frac{15}{4}\right)^2 = \frac{15}{16} = 0.9375$$

Question 4:

Transportation Safety Board of Canada (TSB) report shows that in Canada, the probability of avionic accident because of pilot error is 20%, because of icing is 25%, because of mechanical failure is 30% and the cause of rest of the accidents are unknown. Suppose there are 10 accidents and the type of each accident is independent from accident to accident. Let random variables X, Y, Z and W denote the number of accidents because of pilot error, icing, mechanical and unknown, respectively. Calculate the following:

a) $P(X=3, Y=3, Z=2, W=2)$ and $P(X=8)$

b) $P(X>7|Y=2)$

Solution 4:

4a)

$P_1 = 0.2$ Pilot error → X

$P_2 = 0.25$ Icing → Y

$P_3 = 0.3$ Mechanical failure→ Z

$P_4 = 0.25$ Unknown → W

→ P(X=3,Y=3,Z=2,W=2)=$\frac{10!}{3!3!2!2!} \times 0.2^3 \times 0.25^3 \times 0.3^2 \times 0.25^2$=0.017718→ 1.77%

$$P(X=8) = \binom{10}{8} \times 0.2^8 \times 0.8^2 = 0.00007372 = 7.37 \times 10^{-5} = 0.0073\%$$

4b)

$$P(X>7|Y=2) = \frac{P(X=8|y=2)}{P(Y=2)} + \frac{P(X=7|y=2)}{P(Y=2)}$$

$$= \frac{P(X=8, Y=2, Z=0, W=0)}{P(Y=2)} + \frac{P(X=7, Y=2, Z=1, W=0)}{P(Y=2)} + \frac{P(X=7, Y=2, Z=0, W=1)}{P(Y=2)}$$

$$P(X=8, Y=2, Z=0, W=0) = \frac{10!}{8!\,2!} \times 0.2^8 \times 0.25^2 = 0.0000072$$

$$P(X=7, Y=2, Z=1, W=0) = \frac{10!}{7!\,2!} \times 0.2^7 \times 0.25^2 \times 0.3 = 0.0000864$$

$$P(X=7, Y=2, Z=0, W=1) = \frac{10!}{7!\,2!} \times 0.2^7 \times 0.25^2 \times 0.25 = 0.000072$$

$$P(Y=2) = \binom{10}{2} \times 0.25^2 \times 0.75^8 = 0.281256$$

$$P(X>2|Y=0) = 0.0005881 = 0.02881\%$$

Question 5:

Suppose 10 units of a certain type of processor are tested, and their failure times in hours are

300, 350, 370, 375, 380, 340, 400, 390, 385, and 410. Suppose the failure times are normally distributed.

a) Calculate the sample mean and the sample variance of the failure time of the processor.

b) Determine a 96% upper confident bound on the population mean.

Solution 5:

a). Sample mean: $\bar{x} = \dfrac{\sum_{i=1}^{10} x_i}{10} = 370$

Sample variance: $s^2 = \dfrac{\sum_{i=1}^{10}(x_i - \bar{x})^2}{9} = \dfrac{\sum_{i=1}^{10} x_i^2 - \dfrac{\left(\sum_{i=1}^{10} x_i\right)^2}{10}}{9} = 1050$.

b) The variance is unknown and sample size is not large in this problem. Thus t distribution should be used to determine the confidence interval for the mean. The 96% upper confidence bound is:

$$t_{\alpha,n\text{-}1} = t_{0.04,9} = 1.833*(0.05-0.04)/(0.05-0.025) + 2.262*(0.04-0.025)/(0.05-0.025) = 2.0904$$

$$u = \bar{x} + t_{\alpha,n\text{-}1} \cdot s/\sqrt{n} = 370 + t_{0.04,9} \cdot \sqrt{1050}/\sqrt{10} = 370 + 2.0904 \cdot \sqrt{1050}/\sqrt{10} = 391.42$$

Problem 6:

Suppose $\bar{X}_1$ and $\bar{S}_1^2$ are the sample mean and sample variance from a population with mean μ_1 and variance σ_1^2, and $\bar{X}_2$ and $\bar{S}_2^2$ are the sample mean and sample variance from a second independent population with mean μ_2 and variance σ_2^2. The sample sizes are n_1 and n_2, respectively.

a) Show that $\bar{X}_1 + \bar{X}_2$ is an estimator of $\mu_1 + \mu_2$. Is it biased or unbiased? Justify your answer.

b) Find the standard error of $\bar{X}_1 + \bar{X}_2$.

Solution 6:

a) $\mathrm{E}(\bar{X}_1 + \bar{X}_2) = \mathrm{E}(\bar{X}_1) + \mathrm{E}(\bar{X}_2) = \mu_1 + \mu_2$.

It is an unbiased estimator since $E(\overline{X}_1 + \overline{X}_2) - (\mu_1 + \mu_2) = (\mu_1 + \mu_2) - (\mu_1 + \mu_2) = 0$

b) $\mathrm{s.e.} = \sqrt{\mathrm{V}(\bar{X}_1 + \bar{X}_2)} = \sqrt{\mathrm{V}(\bar{X}_1) + \mathrm{V}(\bar{X}_2) + 2\mathrm{COV}(\bar{X}_1, \bar{X}_2)} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

This standard error can be estimated by using the estimates for the standard deviations of populations 1 and 2. Therefore: Standard Error = $\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

Problem 7:

A manufacturer is interested in the output voltage of a power supply used in a PC. Output voltage is assumed to be normally distributed, with standard deviation 0.25 volt. He/she selects a random sample of 8 units with sample mean of 4.95 volts.

a) Test the hypothesis that the average output voltage is 5 volts against that the average output voltage is not 5 volts. Use a 0.05 of significance and formulate appropriate hypothesis test.
b) Construct a 95% confidence interval on the mean output voltage. Discuss the results of parts (a) and (b).

Solution 7

Part (a):

1) The parameter of interest is the mean output voltage, μ.
2) $H_0 : \mu = 5$
3) $H_1 : \mu \neq 5$
4) $\alpha = 0.05$
5) $Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$
6) Reject H_0 if $z_0 < -z_{\alpha/2}$ where $-z_{0.025} = -1.96$ or $z_0 > z_{\alpha/2}$ where $z_{0.025} = 1.96$
7) For $\bar{x}$= 4.95, $\sigma = 0.25$, and $n = 8$:

$$z_0 = \frac{4.95 - 5}{0.25/\sqrt{8}} = -0.566$$

8) Since $-1.96 < -0.566 < 1.96$, we cannot reject the null hypothesis. In the other words, there is not sufficient evidence to reject that the mean output voltage is equal to 5 at $\alpha = 0.05$.

Part (b):

$$\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}$$
$$4.95 - 1.96(\,0.25/\sqrt{8}) \le \mu \le 4.95 + 1.96(\,0.25/\sqrt{8})$$
$$4.95 - 0.17 \le \mu \le 4.95 + 0.17$$
$$4.78 \le \mu \le 5.12$$

The confidence interval constructed contains the value 5; thus the true mean output voltage could possibly be 5 volts using a 95% level of confidence. Since a two-sided 95% confidence interval is equivalent to a two-sided hypothesis test at $\alpha = 0.05$, the conclusions necessarily must be consistent.

Problem 8:

The weight of a certain product in mg is normally distributed, with standard deviation of σ = 3 mg. A random sample of 36 items is tested and the mean weight is obtained which is equal to 74 mg.

a) To test H0: μ = 75 versus H1: μ > 75, calculate P-value.
b) The probability of type I error is 6%. What is the probability of type II error if the true mean is 77?

Solution 8

Part (a):

For $\bar{x} = 74$, $\sigma = 3$, and $n = 36$:

P-value = $1 - \Phi(Z_0)$ where $Z_0 = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$

$$z_0 = \frac{74-75}{3/\sqrt{36}} = -2$$

$\Phi(Z_0) = 0.023$

P-value = $1 - 0.023 = 0.997$

Part (b):

$\alpha = 0.06$, $n = 36$, then $z_\alpha = 1.555$

Method I:

$$\beta = \Phi\left(z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right) = \Phi\left(1.555 - \frac{(77-75)\sqrt{36}}{3}\right) = \Phi(-2.445) = 0.0077$$

Method II:

Critical value is $75 + 1.555\left(\frac{3}{\sqrt{36}}\right) = 75.778$

$\beta = P(\bar{X} \le 75.778$ when $\mu = 77)$

$$= P\left(Z \le \frac{75.778-77}{3/\sqrt{36}}\right) = P(Z \le -2.444) = \Phi(-2.444) = 0.0077$$