Concordia University

Department of Electrical and Computer Engineering

COEN 231/2 Fall 2001-2002
Discrete Mathematics – Final Exam

Value: 65% of final grade
Time: 180 minutes
Total marks: 65
Write your solutions in the space provided, or on the reverse sides of the pages.

Materials allowed: Three (3) 8½" by 11" crib sheets (both sides)
No computing aids (no calculators).

ID#______________ **Section:** R___U __

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Last name **First name**

READ EACH QUESTION CAREFULLY, AND SHOW YOUR REASONING!
Except where indicated, you may leave your answers in terms of factorials.
There are 12 questions: Plan your time carefully.

1___2____3___4____5____6____7____8____9____10____11____12____

(1) (a) How many different binary numbers can be represented by a string of three hexadecimal characters? (E.g. the hexadecimal string A2B represents the binary number 1010 0010 1011.) Give an exact numerical answer. 7 pts

(b) Consider the hexadecimal string C0D00000. If this string represents a binary arrangement in the 32-bit (single-precision) IEEE 754 format, what decimal number is represented by that binary arrangement?

(c) Multiply the decimal answer to part (b) by 2, and convert it into binary IEEE 754 format.

a) $16^3 = 2^{12} = 4196$

b) \$C = 12 = %1100
\$D = 13 = %1101

\$C0D0000 = 1 | 10000001 | 101000…0
exponent mantissa

Biased exponent = 129.
Remove bias of 127 to obtain actual exponent: 129-127 = 2.
Mantissa: %1.101 = $1 + 2^{-1} + 2^{-3} = 1 + 0.5 + 0.125 = 1.625$
Answer: $1.625 * 2^2 = 6.5$.

c) Exponent increases by one power of 2:

1 | 10000010 | 101000…0

(2) Consider the full adder. <u>6 pts</u>

(a) Write the 8-row truth table for the carry bit of the full adder.
(b) Write the functional representation for the carry bit of the full adder in Boolean Sum of Products format.
(c) Write the functional representation for the carry bit of the full adder in NAND format (suitable for implementation in NAND gates.) (Note that you do not need to draw the gates.)

a)

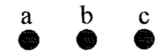
A_i	B_i	C_i	C_{i+1}
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

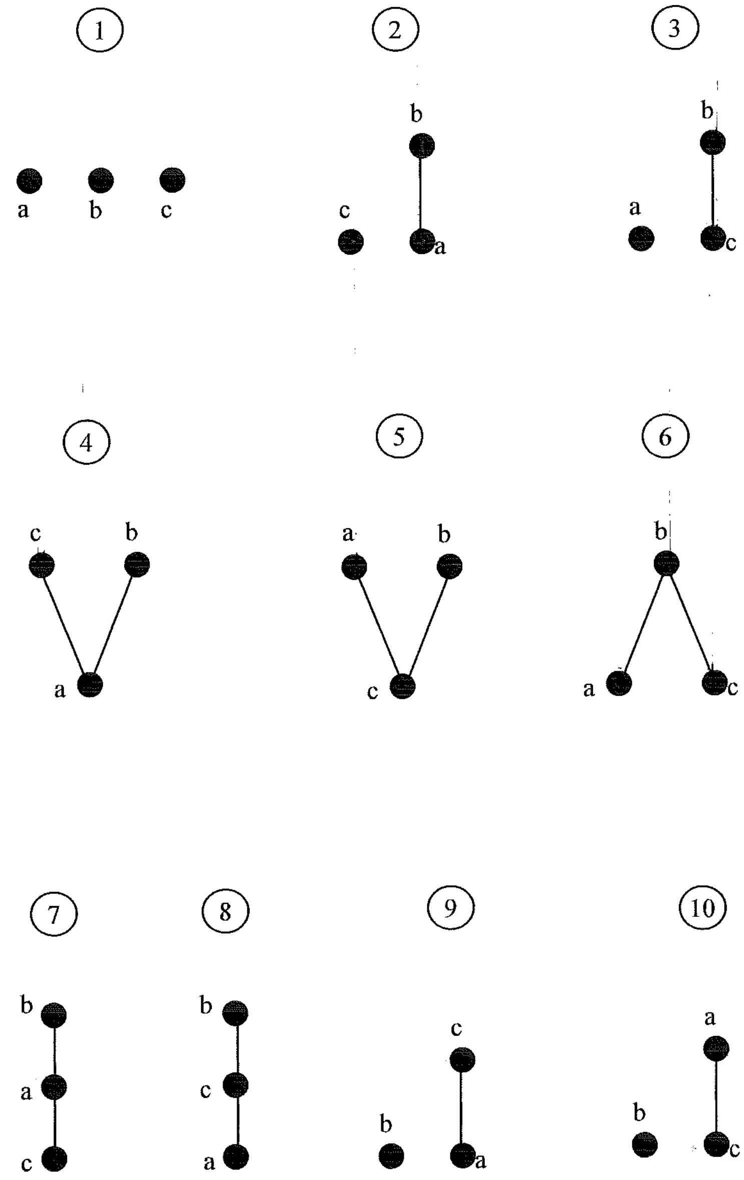
b) $C^{i+1} = A'BC + AB'C + ABC' + ABC$

c) $C^{i+1} = (A'BC + AB'C + ABC' + ABC)''$
$= [(A'BC)'(AB'C)'(ABC')'(ABC)']'$

(3) Consider partial orderings constructed on the set {a, b, c}. **Draw** all possible Hasse diagrams for these partial orderings, where the element "b" is a maximal element. <u>6 pts</u>

Hints: There are ten small Hasse diagrams to draw. The relation {(a,a),(b,b),(c,c)}, having Hasse diagram at right, above, is one of them.





(4) "All correct arguments are logically valid. Some logically invalid arguments sound reasonable. We conclude that some reasonable sounding arguments are incorrect." 5 pts

a) Define the primitive open statements
b) Construct the argument
c) Prove it.

Hint: The universe consists of arguments.

a) c(x): x is a correct argument.
   l(x): x is a logically valid argument.
   r(x): x is a reasonably sounding argument.

b)

$$\begin{array}{l} \forall x[\ c(x) \rightarrow l(x)\ ] \\ \exists x[\ l'(x)\ r(x)\ ] \\ \hline \therefore \exists x[\ r(x)\ c'(x)\ ] \end{array}$$

c) Using universal and existential specification, we get:

$$\begin{array}{ll} c(e) \rightarrow l(e) & \} \Rightarrow c(e) \\ l'(e) & \} \text{ (Modus Tollens)} \\ r(e) & \\ \hline \therefore r(e)\ c'(e) & \end{array}$$

We thus have :

$$\begin{array}{ll} c'(e) & \} \ \ r(e)\ c'(e) \\ r(e) & \} \\ \hline \therefore r(e)\ c'(e) & \end{array}$$

The argument is proven by direct verification: the conclusion was inferred by the premises.

(5) Let P and Q and R be subsets of a universal set U. <u>6 pts</u>

Prove, using inference format and the language of logic with quantifiers, that $[R \subseteq [P' \cap Q]] \Rightarrow [[P \cap R] \Leftrightarrow \varnothing]$.

Proof by an example is NOT acceptable.

$$\frac{\forall x[\ r(x) \rightarrow p'(x)q(x)\ ]}{\therefore [\exists x\ p(x)r(x)]' \Leftrightarrow [\forall x\ p'(x) + r'(x)\ ] \Leftrightarrow \forall[\ r(x) \rightarrow p'(x)\ ]}$$

<u>Now</u>: Using universal specification:

$$\frac{r(a) \rightarrow p'(a)q(a)}{\therefore r(a) \rightarrow p'(a)}$$

The expression $r(a) \rightarrow p'(a)q(a) \quad \Leftrightarrow$
$[r(a) \rightarrow p'(a)]\ [\ r(a) \rightarrow q(a)]$

We now have:

$$\frac{\begin{array}{l} r(a) \rightarrow p'(a) \\ r(a) \rightarrow q(a) \end{array}}{\therefore r(a) \rightarrow p'(a)}$$

Now applying the rule of conjunctive simplification, we have proven by direct validation, that the conclusion is inferred by the premises.

(6) Consider the following "proofs". We are to prove the hypothesis: <u>6 pts</u>

$\forall x \in \mathbf{R}\ \forall y \neq x\ [x^2 > y^2] \Rightarrow [x > y]$

a) Identify what is wrong with each of the following proofs

I. $16 > 9$. $16 = 4^2$ and $9 = 3^2$. Now $4 > 3$ which is also true. We could repeat this for other real numbers. So the argument is proved.

Proof by example. Cow counting.

II. Let us assume that $\forall x \in \mathbf{R}\ \forall y \neq x\ [x^2 > y^2]$ is true. Using universal specification, we get $[a^2 > b^2]$. This can be rewritten as $a*a > b*b$. We divide LHS by a and RHS by b to yield $a > b$. Using universal generalization, we get the result.

(1) In general, we cannot divide the LHS + RHS by different numbers.

b) Construct a counterexample which disproves the hypothesis.

Counterexample:
$x = -5$, $y = -3$

$x^2 = (-5)^2 = 25 > y^2 = (-3)^2 = 9$

But $-5 < -3$

Hence, the statement $[x^2 > y^2] \Rightarrow [x > y]$ is false, for this example.

(7) "Within the set of Concordia students, if any COEN231 student has not studied all of the assignments, then such student(s) will have difficulty with at least one of the tests." 6 pts

(a) Model this English sentence using the language of logic with multiple quantifiers. Your final answer should include clear definitions of your open statements, and all relevant sets. Hint: You will need three different variables, x,y,z, to handle students, assignments and tests.

(b) Find the negation of the above statement, both symbolically and in good English.

We can rewrite the statement as follows:

"Within the universe of Concordia students, if any student is in COEN 231, then if there exists at least on assignment which the student has not studied, then there exists at least one test with which he/she will have difficulty."

a)

coen(x) :	student x is in COEN 231.
stud(x,y) :	student x has studied assignment y.
diff(x,z) :	student x has difficulty with test z.

The statement, expressed in quantified logic notation:

$\forall x\{ \text{coen}(x) \rightarrow [\ \exists y\ \text{stud}'(x,y) \rightarrow \exists z\ \text{diff}(x,z)\ ]\ \}$

Rewriting the statement using basic connectives :

$\forall x\{ \text{coen}'(x) + \forall y\ \text{stud}(x,y) + \exists z\ \text{diff}(x,z) \} \quad \Leftrightarrow$

$\forall x \forall y \forall z\ \{ \text{coen}'(x) + \text{stud}(x,y) + \text{diff}(x,z) \} \quad \Leftrightarrow$

$\forall x \forall y \forall z\ \{ [\text{coen}(x)\ \text{stud}'(x,y)]' + \text{diff}(x,z) \} \quad \Leftrightarrow$

$\forall x \forall y \forall z\ \{ [\text{coen}(x)\ \text{stud}'(x,y)] \rightarrow \text{diff}(x,z) \} \quad \Leftrightarrow$

b)

The negation of the original statement is :

$\exists x \exists y \exists z\{ \text{coen}(x)\ \text{stud}'(x,y)\ \text{diff}'(x,z) \}$

The negation, expressed in English :

"There exists at least one COEN 231 student who did not study at least one test, but who had no difficulty with any test."

(8) Prove by induction that for all integers $n \geq 0$, $((n^3 - 7n + 3)/3) \in \mathbf{Z}$, namely that $n^3 - 7n + 3$ is divisible by 3 (with no remainder). <u>5 pts</u>

<u>Base Step</u>: n=0, $3/3 \in Z$

<u>Induction Step</u>:

Assume $\frac{k^3 - 7k + 3}{3} \in Z$ and

Prove that $\frac{(k+1)^3 - 7(k+1) + 3}{3} \in Z$

Proof:

$1/3 \{ k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 \}$

$= 1/3 \{ (k3 - 7k + 3) + (3k2 + 3k - 6) \}$
$= 1/3 \{ \quad a \quad + \quad b \quad \}$

Now $a/3 \in Z$ by hopothesis.

$\frac{3k^2 + 3k - 6}{3} = k^2 + k - 2 \in Z$ $(\forall k \in Z)$

QED

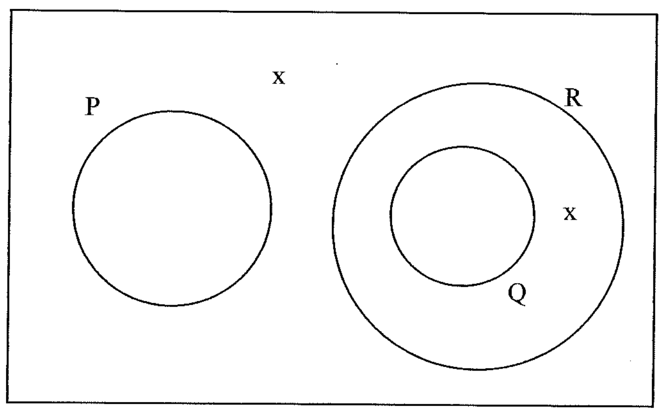
(9) A given universe contains three sets P, Q and R. Note that three sets define eight disjoint sets, each of which may or may not be populated.

(a) Construct a Venn Diagram such that the following two constraints are satisfied. Use x's to indicate which of the disjoint sets must not be empty. <u>2 pts</u>

- $Q \subset R$
- $P \subset R'$

(b) How many ways are there to place eight identical elements into this universe, such that the two constraints described in part (a) are satisfied? (Give an exact numerical answer.) <u>3 pts</u>

a)



b)

6 elements left to place in four boxes (disjoint subsets).

$$C(6+4-1,6) = C(9,6) = \frac{9!}{6!3!} = \frac{9\cdot 8\cdot 7}{6} = 3\cdot 4\cdot 7 = 84$$

↳3! = 3rd

(10) A market survey of 50 people concerning their preference for three brands of cookies determines that: **5 pts**
47 people like at least one brand;
26 people like brand A, 22 like brand B, and 34 like brand C;
11 people like brands A and B, 19 like brands A and C, 14 like brands B and C.
How many people like all three brands?

Use the principle of inclusion/exclusion.

$|A \cup B \cup C| = 47.$
$|A| = 26,\quad |B| = 22,\quad |C| = 34,$
$|A \cap B| = 11,\quad |A \cap C| = 19,\quad |B \cap C| = 14,\quad |A \cap B \cap C| = ?$

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ 47 &= 26 + 22 + 34 - 11 - 19 - 14 + ? \\ &= 82 - 14 + ? \\ &= 38 + ? \end{aligned}$$

$\therefore |A \cap B \cap C| = 9$

(11) Given a set containing 5 distinct dogs and 7 distinct cats, how many ways are there to choose an unordered group of five animals from this set, containing a minimum of 1 dog? (Hint: You might calculate and use the number of ways to choose an unordered group of five from this set, containing no dogs.) You may leave your answer in terms of factorials. **3 pts**

Total = [# of ways of choosing 5 from 12]– [# of ways of having no dogs]

$= C(12,5) - C(7,5)C(5,0)$

(12) There are five distinct pigeons resting in a group of n distinct pigeonholes. No pigeonhole is empty, and no pigeonhole contains more than two pigeons. <u>5 pts</u>

(a) List the possibilities for n, the number of pigeonholes.

(b) If there are 90 ways to arrange the pigeons in the pigeonholes (where the order of the pigeons within a particular pigeonhole does not matter), find the number n (how many pigeonholes are there?) **Show your reasoning clearly.**

Note: This problem is not about the pigeonhole principle.

a)

2 holes not possible since one must contain at least 3.

3 holes is possible:

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x x
x x x
```

4 holes is possible:

```
x
x x x x
```

5 holes is possible:

```
x x x x x
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b)

Try 3 holes:

$$(5 \cdot 4 \cdot 3) \quad x \quad C(3,2) \quad x \quad (2 \cdot 1) \quad x \quad \frac{1}{2!} x \frac{1}{2!}$$

- $(5 \cdot 4 \cdot 3)$: Placing pigeons In holes
- $C(3,2)$: Choose 2 holes
- $(2 \cdot 1)$: Placing last 2 pigeons
- $\frac{1}{2!} x \frac{1}{2!}$: Order in the holes doesn't matter

$$= \frac{5 \cdot 4 \cdot 3 \text{ x } 3 \text{ x } 2}{4} = 5 \cdot 3 \cdot 3 \cdot 2 = 90$$