Fall Term, 2015

Name: ________________ **Last Name:** ________________

Section:

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 121 – Introduction to Mathematical Proofs

Midterm 2 – November 20, 2015

SOLUTIONS

Instructors: *Dr. P. Zhang*

Time Allowed: *80 minutes*

Total Value: *80 marks*

Number of Pages: *4 plus cover page*

Instructions:

No calculators are allowed. *No other aids are allowed.*

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the page.***

Show all your work. ***Insufficient justification will result in a loss of marks.***

Student Number: ____________________

[*14 marks*] 1. Consider the set U of **170 students** in the MA121 class with the following data:

40 students are **also** taking MA122;

The subset B of U consists of 25 students who are **also** taking MA103;

The subset C of U consists of 10 students who are **also** taking MA104;

20 students are taking **both** MA122 and MA103;

MA103 and MA104 cannot be taken at the same time;

5 students are taking **both** MA122 and MA104.

(a) Fill the blanks:

The number of elements of the set $A = \{x \in U : x \text{ studies MA122}\}$ is __**40**__ .

The students of MA121 who are also taking both MA103 and MA104 form the set __$\emptyset$__ .

The number of elements of the set $A \cap B \cap C$ is $|A \cap B \cap C| =$ __**0**__ .

In parts (b), (c), (d) and (e) below, either *fill the blanks* **or** *circle the correct answers***. The symbols of sets are introduced above**.

(b) The number of students who are taking MA121 and at least another mathematics course at the same time form the set $A \cup$ _$\mathbf{B} \cup \mathbf{C}$__ . The number of elements of this set that can be computed by the formula

$$|A| + \_\_|\mathbf{B}|\_\_ + |C| \_\_-\_\_ |A \cap B| - |A \cap C| - | \_\_|\mathbf{B} \cap \mathbf{C}|\_\_ | \_\_+\_\_ |A \cap B \cap C|.$$

The number of students in this class who are also taking at least another mathematics course is

(i) 75

(ii) 50

(iii) 70

(c) The students in this class who are also taking **two more** mathematics courses form the set

(i) $A \cup B \cup C$

(ii) $(A \cap B) \cup (A \cap C) \cup C$

(iii) $A \cap (B \cup C)$

The number of elements of this set can be computed by the formula

$$|A \cap B| + |A \cap C| - | \_\_\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}\_\_ |,$$

which is

(i) 30

(ii) 25

(iii) 20

(d) The number of students in this class who are also taking **exactly one of** the three courses MA122, MA103 and MA104 is

(i) 15

(ii) 50

(iii) 25

(e) The number of students in this class who are **not** taking any other mathematics course is

(i) 145

(ii) 120

(iii) 95

Over

2. Let m and n be **nonnegative** integers.

[*3 marks*] (a) Fill the blanks so that the following identity holds **for each** $n \geq 0$:

$$\binom{m+n+2}{\_\_\mathbf{m+1}\_\_} = \binom{m+n+1}{\_\_\mathbf{m}\_\_} + \binom{\_\_\mathbf{m+n+1}\_\_}{m+1}$$

[*5 marks*] (b) Write the sum

$$\binom{m}{m} + \binom{m+1}{m} + \binom{m+2}{m} + \cdots + \binom{m+n}{m} = \binom{m+n+1}{m+1}$$

in **sigma notation** by filling the blanks:

$$\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{n}} \binom{\_\_\mathbf{m+j}\_\_}{\_\_\mathbf{m}\_\_} = \binom{m+n+1}{m+1}$$

[*8 marks*] (c) Use the identity in part (a) and the Principle of **Mathematical Induction** to **prove that the identity in part (b) holds** for **all integers** $n \geq 0$.

Proof. The base case here is for $n = 0$.

LHS$= \sum_{j=0}^{0} \binom{m+j}{m} = \binom{m+0}{m} = \binom{m}{m} = 1.$

RHS$= \binom{m+0+1}{m+1} = \binom{m+1}{m+1} = 1.$ The identity holds.

Assume that the identity holds for an integer $k \geq 0$; that is,

$$\sum_{j=0}^{k} \binom{m+j}{m} = \binom{m}{m} + \binom{m+1}{m} + \binom{m+2}{m} + \cdots + \binom{m+k}{m} = \binom{m+k+1}{m+1}.$$

Now for $n = k+1$,

$$\begin{aligned}
LHS &= \sum_{j=0}^{k+1} \binom{m+j}{m} = \Big(\sum_{j=0}^{k} \binom{m+j}{m}\Big) + \binom{m+(k+1)}{m} \\
&= \Big(\binom{m}{m} + \binom{m+1}{m} + \binom{m+2}{m} + \cdots + \binom{m+k}{m}\Big) + \binom{m+k+1}{m} \\
&= \binom{m+k+1}{m+1} + \binom{m+k+1}{m} \quad \text{(by the inductive hypothesis above)} \\
&= \binom{m+k+2}{m+1} \quad \text{(by the formula in part (a) above)}
\end{aligned}$$

$RHS = \binom{m+(k+1)+1}{m+1} = \binom{m+k+2}{m+1} = RHS.$

Therefore the identity holds for all integers $n \geq 0$.

3. Let $A = \{a_1, a_2, a_3\}$ and B be a set with $|B| = 5$.

[*1 marks*] (a) Fill the blanks:

A function $f : A \to B$ is one-to-one if and only if $a_i \neq a_j$ in A implies that $f(a_i)$ __$\neq$__ $f(a_j)$ in B.

[*5 marks*] (b) Apply the **Multiplication Principle** to prove that the number of **distinct one-to-one functions** from A into B equals 60.

Proof. If $f : A \to B$ is a function, then $f(a_1)$ may be any one of the five elements of B.

Because $f : A \to B$ is one-to-one, $f(a_2) \neq f(a_1)$, and hence $f(a_2)$ may only be one of the four remaining elements once $f(a_1)$ is fixed.

After $f(a_1)$ and $f(a_2)$ are fixed, $f(a_3)$ may only take one of the remaining three values in B since $f(a_3) \neq f(a_1)$ and $f(a_3) \neq f(a_2)$.

Therefore there are

$$5 \times 4 \times 3 = 60$$

distinct one-to-one functions from A into B by the Multiplication Principle.

4. **Prove** or **disprove** each of the following statements **using the specified method**.

[*5 marks*] (a) Use the method of **indirect proof of implication by contradiction** to prove the statement:

If z_1 and z_2 are complex numbers, then $|z_1 - z_2| \geq |z_1| - |z_2|$.

[Hint: Find a contradiction to the Triangle Inequality: $|w_1 + w_2| \leq |w_1| + |w_2|$ by setting $w_1 = z_1 - z_2$.]

Proof. Asumme that z_1 and z_2 are complex numbers, but $|z_1 - z_2| < |z_1| - |z_2|$.

It follows that $|z_1| > |z_1 - z_2| + |z_2|$.

Let $w_1 = z_1 - z_2$ and $w_2 = z_2$, both complex numbers.

Then $|w_1 + w_2| = |z_1 - z_2 + z_2| = |z_1| > |z_1 - z_2| + |z_2| = |w_1| + |w_2|$.

This is a contradiction to the Triangle Inequality: $|w_1 + w_2| \leq |w_1| + |w_2|$.

Therefore the statement is true.

[*5 marks*] (b) Let A be a nonempty set with $|A| = n$, and let $f : A \to A$ be a function. Apply the **Pigeonhole Principle** to **prove the contrapositive** of the statement:

If f is a one-to-one function, then it is also an onto function.

Proof. We shall prove that if $f : A \to A$ is NOT onto, then f is NOT one-to-one.

Suppose that f is not onto. Then the set of outputs of f is a subset of A with less than n elements.

But the set of inputs of f, i.e., the domain of f is A with n elements. By the Pigeonhole Principle, there must be (at least) two distinct elements a_i and a_j (a pair of pigeons!) with $f(a_i) = f(a_j)$.

This implies that f is not one-to-one.

[*5 marks*] (c) Let A and B be subsets of the same universal set and denote by $A \triangle B$ the symmetric difference of A and B. Apply De Morgan's laws to **prove directly** the statement:

If $A^c \cup B^c = A^c \cap B^c$, then $A \triangle B = \emptyset$.

Proof. Suppose that $A^c \cup B^c = A^c \cap B^c$.

By the De Morgan's laws,

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B \text{ and } (A^c \cap B^c)^c = (A^c)^c \cup (B^c)^c = A \cup B.$$

It follows from the assumption that $A^c \cup B^c = A^c \cap B^c$ that

$$A \cap B = (A^c \cup B^c)^c = (A^c \cap B^c)^c = A \cup B.$$

Therefore

$$A \triangle B = (A \cup B) - (A \cap B) = \emptyset.$$

[*5 marks*] (d) Demonstrate a **counterexample** to **disprove** the statement:

The complex number $z = 3 + 4i$ is the only one with $|z| = 5$ such that $z - \bar{z} = 8i$.

Solution. Let $z = -3 + 4i$.

Then $\bar{z} = -3 - 4i$ and hence $z - \bar{z} = -3 + 4i - (-3 - 4i) = 8i$.

Also $|z| = \sqrt{(-3)^2 + 4^2} = 5$.

This is a counterexample to the statement.

[*8 marks*] 5. Find the **constant term** (that is, the coefficient of x^0) in the expansion of $\left(x - \frac{1}{x^2}\right)^9$. Show **all steps** of your work.

Solution. By the Binomial Theorem, the k-th term of the expansion is

$$\binom{9}{k} x^{9-k}\left(-\frac{1}{x^2}\right)^k = \binom{9}{k}(-1)^k x^{9-k}x^{-2k} = \binom{9}{k}(-1)^k x^{9-3k}.$$

The term is a constant a if and only if $9 - 3k = 0$; that is, $k = 3$.

Thus

$$a = \binom{9}{k}(-1)^k = -\frac{9!}{3!6!} = \frac{9\times 8\times 7}{3\times 2\times 1} = -3\times 4\times 7 = -84.$$

[*8 marks*] 6. Let A and B be subsets of the universal set U. Denote by A^c and B^c the complements of A and B in U respectively. Prove that, as subsets of the product set $U\times U$,

$$(A\times A^c)\cap(B^c\times B) = (A\times B)\cap(B^c\times A^c).$$

Proof. Let $(x,y)\in U\times U$. Because

$$\begin{aligned}
&(x,y)\in(A\times A^c)\cap(B^c\times B)\\
&\quad\text{iff}\ \ (x,y)\in A\times A^c\ \ \text{and}\ \ (x,y)\in(B^c\times B)\\
&\quad\text{iff}\ \ (x\in A\text{ and }y\in A^c)\text{ and }(x\in B^c\text{ and }y\in B)\\
&\quad\text{iff}\ \ (x\in A\text{ and }y\in B)\text{ and }(x\in B^c\text{ and }y\in A^c)\\
&\quad\text{iff}\ \ (x,y)\in A\times B\ \ \text{and}\ \ (x,y)\in B^c\times A^c\\
&\quad\text{iff}\ \ (x,y)\in(A\times B)\cap(B^c\times A^c),
\end{aligned}$$

It follows that $(A\times A^c)\cap(B^c\times B) = (A\times B)\cap(B^c\times A^c)$.

Alternatively,

$$\begin{aligned}
&(A\times A^c)\cap(B^c\times B)\\
&=\{(x,y)\in U\times U|(x,y)\in A\times A^c\ \textit{and}\ (x,y)\in B^c\times B\}\\
&=\{(x,y)\in U\times U|(x\in A\ \textit{and}\ y\in A^c)\ \textit{and}\ (x\in B^c\ \textit{and}\ y\in B)\}\\
&=\{(x,y)\in U\times U|(x\in A\ \textit{and}\ y\in B)\ \textit{and}\ (x\in B^c\ \textit{and}\ y\in A^c)\}\\
&=(A\times B)\cap(B^c\times A^c).
\end{aligned}$$

[*8 marks*] 7. Simplify $\frac{1-i}{1+i} - \frac{1+i}{2i}$. Express your answer in **standard form**, $a+bi$.

Solution.

$$\begin{aligned}
&\frac{1-i}{1+i} - \frac{1+i}{2i}\\
&=\frac{1-i}{1+i}\cdot\frac{1-i}{1-i} - \frac{1+i}{2i}\cdot\frac{-2i}{-2i}\\
&=\frac{1+i^2-2i}{1-i^2} - \frac{-2i-2i^2}{-4i^2}\\
&=\frac{-2i}{2} - \frac{-2i+2}{4}\\
&=-i+\frac{1}{2}i-\frac{1}{2}\\
&=-\frac{1}{2}-\frac{1}{2}i
\end{aligned}$$

Alternatively,

$$\begin{aligned}
&\frac{1-i}{1+i} - \frac{1+i}{2i}\\
&=\frac{2i(1-i)-(1+i)^2}{2i(1+i)}\\
&=\frac{2i-2i^2-(1+i^2+2i)}{2i+2i^2} = \frac{2i+2-1+1-2i}{-2+2i} = \frac{1}{-1+i} = \frac{-1-i}{(-1+i)(-1-i)}\\
&=\frac{-1-i}{2} = -\frac{1}{2}-\frac{1}{2}i
\end{aligned}$$