CARLETON UNIVERSITY

MOCK FINAL EXAMINATION
MATH 2004
Fall 2014

DURATION: 3 HOURS

Department Name and Course Number: School of Mathematics and Statistics, MATH 2004 A, B, C, D.
Course Instructor(s): Dr. A.B. Mingarelli (Sect. A), Dr. R. Cova (Sect. B, C, D)

AUTHORIZED MEMORANDA
STUDIO 56 SCIENTIFIC CALCULATOR ONLY AS PER COURSE OUTLINE.

In addition to the examination paper students will require an EXAMINATION BOOKLET, and a SCANTRON SHEET. This exam may be released to the Library.

1. Please verify that you are in possession of a SCANTRON FORM
2. Please fill in your COURSE CODE (e.g., MATH 2004) and COURSE SECTION (e.g., A, B, C, D), as per above list of instructors; YOUR NAME and YOUR STUDENT NUMBER where required on the Scantron form AND on this examination.
3. **The entire examination consists of 5 pages and two parts, A and B, and is marked out of a total of 80**, that is, 40% of your final mark. Part A consists of 12 multiple choice questions each worth 3 marks. **Please fill in only one answer on your Scantron sheets with a pencil** as there is only one answer to any given question. Circling two or more answers to any question invalidates that question (*i.e.,* you get 0 marks for that question). Part B consists of 6 traditional type questions (with explanations required), for a total of 44 marks for that section. **Both Part A and B must be submitted along with the scantron sheet, so do not detach nor unstaple this examination. Do not submit rough work. Do not submit examination booklet**.

Print Name :

Student Number:

Section (either A, B, C, or D):

- **A1.** Find the slope of the tangent line to $r = 3\sin\theta$ at $\theta = \frac{\pi}{2}$.

  (a) 0 (b) $\sqrt{3}$ (c) 2 (d) 3

  Solution: (a)

- **A2.** Let θ be the angle between the two planes $x + 2y - 2z - 3 = 0$ and $2x + 2y - z - 5 = 0$. Find $\cos\theta$.

  (a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{8}{9}$ (d) $\frac{9}{8}$

  Solution: (c)

- **A3.** Find the area of the triangle with vertices $P(1, 2, 3)$, $Q(-3, 0, 1)$, and $R(2, 4, 5)$.

  (a) $6\sqrt{2}$ (b) $6\sqrt{3}$ (c) 12 (d) $3\sqrt{2}$

  Solution: (d)

Print Name and Student Number: ______________________________

- **A4.** Given the line $L : \mathbf{r} = (1, 2, 3) + t(3, 2, -1)$. Let Π be the plane through the point $(1, 1, 3)$ and perpendicular to the line L. Which of the following is a point on the plane Π ?

  (a) $(0, 0, -1)$ (b) $(0, 0, -2)$ (c) $(0, 0, -3)$ (d) $(0, 0, -4)$

  Solution: (b)

- **A5.** Find all the critical points of $h(x, y) = y^2 - 2y - y\sqrt{x} + \frac{x}{2}$ where x, y are real, $x \geq 0$.

  (a) $(4, 2)$, $(0, 0)$ (b) $(0, 0)$ only (c) $(2, 4)$, $(0, 0)$ (d) $(4, 2)$

  Solution: (a)

- **A6.** Evaluate the double integral $\iint\limits_{\mathcal{R}} e^{-x} \sin y \, dx \, dy$, where $\mathcal{R} = \{(x, y) \mid 0 \leq x \leq \ln 2, \; 0 \leq y \leq \pi/2\}$.

  (a) $1 - e$ (b) $\ln 2 - 1$ (c) 0 (d) $\frac{1}{2}$

  Solution: (d)

- **A9**. Which one of the following integrals is equal to the double integral $\iint\limits_{\mathcal{R}} 2xy \; dA$, under the change of variables $x = 3u, y = 4v$ where $\mathcal{R}$ is the elliptical region $16x^2 + 9y^2 \leq 144$.

  (a) $\iint\limits_{\mathcal{S}} 12uv \; dA$ (b) $\iint\limits_{\mathcal{S}} 288uv \; dA$, (c) $\iint\limits_{\mathcal{S}} 8(u^2 + v^2) \; dA$ (d) $\iint\limits_{\mathcal{S}} 12u^2v^2 \; dA$

  where $\mathcal{S} = \{(u, v) : u^2 + v^2 \leq 1\}$.

  Solution: (b)

Print Name and Student Number: ______________________________

- **A10.** Evaluate the line integral $\int_{\mathcal{C}} (6xy^2 - y)\, dx + (6x^2y - 3x^2y^2)\, dy$ where $\mathcal{C}$ is the straight line from the point $(1, 2)$ to $(3, 4)$.

  (a) 236 (b) 0 (c) 188 (d) none of them

  Solution: (d)

- **A11.** Let $\mathbf{F}(x, y, z) = 7x\mathbf{i} - z\mathbf{k}$. Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ dS$ where $\mathcal{S}$ is the surface of the sphere $x^2 + y^2 + z^2 = 4$ and $\mathbf{n}$ is an outer normal unit vector to $\mathcal{S}$.

  (a) 24π (b) 16π (c) 0 (d) 64π

  Solution: (d)

End of Part A

Print Name and Student Number: ______________________________

<u>PART B</u>: Do All Four (6) Questions for a total of 44 marks out of a maximum of 80.
Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **B1.** [8 marks] Using the method of Lagrange multipliers determine the local extrema and saddle points (if any) of $f(x, y, z) = x + 2y - 2z$ subject to the constraint $x^2 + 2y^2 + 4z^2 = 1$.

  Solution: $\nabla f = \lambda \nabla g,\ g = x^2 + 2y^2 + 4z^2 \longrightarrow \lambda = \pm 1$. The critical points are then $CP1(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4})$, $CP2(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4})$. One finds that $f(CP1) = 2$ and $f(CP2) = -2$ are the local maximum and minimum, respectively.

- **B2.** [7 marks] By changing to polar coordinates evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2)\, dx\, dy$.

  Solution: $\int_0^{\pi/2} \int_0^1 \cos(r^2)\ r dr\, d\theta = \frac{\pi}{4} \sin(1)$.

Print Name and Student Number: ______________________

- **B3.** [7 marks] The parametric curve C in the plane is given by $x = 2t^2$, $y = 1 + t^3$, where $0 \le t \le 1$. Find the length of the curve C.

Solution: We have $x'(t) = 4t, y'(t) = 3t^2$. The length is

$$\begin{aligned} L &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_0^1 \sqrt{16t^2 + 9t^4}\, dt = \int_0^1 t\sqrt{16 + 9t^2}\, dt \\ &= \frac{1}{18}\int_{16}^{25} \sqrt{u}\, du, \quad u = 16 + 9t^2, \\ &= \frac{1}{27} u^{3/2}\Big|_{16}^{25} = \frac{61}{27}. \end{aligned}$$

- **B4.** [7 marks] Find the area of the region enclosed by the cardioid $r = 1 + \sin\theta$ and the circle $r = 3\sin\theta$.

Solution: The intersection points of the two curves are given by $1 + \sin\theta = 3\sin\theta$, from which we get $\sin\theta = 1/2$ or $\theta = \pi/6$ or $5\pi/6$. The area is then given by

$$\begin{aligned} A &= \int_\alpha^\beta \frac{1}{2}(r_0^2 - r_1^2)\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin^2\theta - (1 + \sin\theta)^2)\, d\theta \\ &= \pi. \end{aligned}$$

Print Name and Student Number: ______________________________

- **B5.** [7 marks] Let $\mathbf{F}(x,y,z) = \dfrac{1}{yz}\,\mathbf{i} - \dfrac{x}{y^2z}\,\mathbf{j} - \dfrac{x}{yz^2}\,\mathbf{k}$.

a) Compute curl $\mathbf{F}$.
b) Find a scalar field $f(x,y,z)$ such that $\nabla f = \mathbf{F}$. If no such scalar field exists, explain why this is not possible.

Solution:

a) curl $\mathbf{F} = \mathbf{0}$.

b) $f(x,y,z) = \dfrac{x}{yz}$.

- **B6.** [8 marks] Let $\mathbf{F}(x,y,z) = -5y\,\mathbf{i} + 4x\,\mathbf{j} + z\,\mathbf{k}$ and let $\mathcal{S}$ be the surface bounded by the circle $x^2 + y^2 = 4$ and $z = 1$. Evaluate the surface integral $\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS$ using any method. Here $\mathbf{n}$ is an outer normal unit vector to $\mathcal{S}$.

Solution:

Direct calculation gives that curl $\mathbf{F} = 9\mathbf{k}$. Inn additon, because the normal lies on the plane $z = 1$ it is clear that $\mathbf{n} = \mathbf{k}$. Hence curl $\mathbf{F} \cdot \mathbf{n} = 9$. Next the region $\mathcal{R}$ in the $uv-$plane is a circle of radius 2. It follows that

$$\begin{aligned} \iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS &= 9 \iint_{\mathcal{R}} du\, dv \\ &= 9 \cdot \text{Area of } \mathcal{R} \\ &= 9\pi 2^2 = 36\pi. \end{aligned}$$