Fall Term, 2015

Name: Solutions

Lab Time: ____________________

Class Section: A-10:30AM B-12:30PM (circle one)

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 103 – Calculus I

Midterm – November 9, 2015, 8:30PM or 9:50PM

Instructor:

Dr. Chester Weatherby

Time Allowed: *80 minutes*

Total Value: *60 marks*

Number of Pages: *6 plus 2-sided cover page*

Instructions:

The CASIO FX-300MS Plus calculator is permitted. No other aids are allowed.

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. **Please note that questions are printed on both sides of the test pages.** *If you require more space on any question, make a note and continue on page 6.*

Show all your work. Insufficient justification will result in a loss of marks. Partial credit will be awarded according to the completeness of your work.

Student Number: ______________

For grading purposes (leave blank)

Question:	Q1	Q2	Q3	Q4	Q5	Q6	Q7	**Total**
Grade:								
Out of:	10	5	8	12	5	7	13	60

[*10 marks*] **1.** Without using L'Hôpital's Rule, evaluate the following limits if they exist:

(a) $\lim_{x\to -1} \frac{x+1}{x^2-2x-3}$ $= \lim_{x\to -1} \frac{x+1}{(x+1)(x-3)} = -\frac{1}{4}$

2.5 each

(b) $\lim_{x\to 5^-} \frac{|x-5|}{2x-10}$ $x<5 \to x-5<0$ so $|x-5| = -(x-5)$

$= \lim_{x\to 5^-} \frac{-(x-5)}{2(x-5)} = -\frac{1}{2}$

(c) $\lim_{x\to 0} \left(\frac{x}{\sqrt{x+25}-5}\right) \frac{(\sqrt{x+25}+5)}{(\sqrt{x+25}+5)}$

$= \lim_{x\to 0} \frac{x(\sqrt{x+25}+5)}{x+25-25}$

$= \lim_{x\to 0} \sqrt{x+25}+5 = \sqrt{25}+5 = 10$

(d) $\lim_{x\to\infty} \frac{\sqrt{4x^4+3}}{x^2-5x^{3/2}+2}$ $\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$

~~Note: x→0 so x~~
$x^2 = \sqrt{x^4}$ (no issue with signs)

$= \lim_{x\to\infty} \frac{\sqrt{\frac{4x^4}{x^4}+\frac{3}{x^4}}}{1-\frac{5}{\sqrt{x}}+\frac{2}{x^2}} = 2$

[5 marks] **2.** $f(x)$ is defined as $f(x) = \begin{cases} e^{Ax} & \text{when } x > 0 \\ x^2 + B & \text{when } x \le 0 \end{cases}$

For what values of A and B will $f(x)$ be continuous and differentiable everywhere?

continuity gives $\lim_{x \to 0} f(x) = f(0)$ so LHL = RHL →

LHL: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + B = B$

2.5

RHL: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{Ax} = e^0 = 1$

LHL = RHL, so $B = 1$

Differentiability gives left slope equals right slope

2.5

Thus Ae^{Ax} @ $0 \to A$; $2x$ @ $0 \to 0$ } $A = 0$

[3 marks] **3.** (a) State the limit definition of the derivative.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ OR $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

[5 marks] (b) Using the limit definition of the derivative, compute the derivative for the function: $f(x) = x^2 - 2x + 3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) + 3 - [x^2 - 2x + 3]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 2)}{h} = 2x - 2$$

[*12 marks*] **4.** For each of following, find $y' = \dfrac{dy}{dx}$ using any method you wish. Leave your answer in terms of x or x and y. Do **NOT** spend time simplifying your answers.

(a) $y = x^2 5^x$

$$y' = 2x5^x + x^2 5^x \ln(5)$$

3 each

(b) $y = \dfrac{1+x^2}{2\tan(x)}$

$$y' = \frac{2x(2\tan(x)) - (1+x^2)(2\sec^2(x))}{4\tan^2(x)}$$

(c) $y = (\sin(x))^{1/x}$ Need logarithmic differentiation

$$\ln(y) = \ln\left((\sin(x))^{1/x}\right) = \frac{\ln(\sin(x))}{x}$$

imp diff ↳ $$\frac{1}{y}\frac{dy}{dx} = \frac{\frac{1}{\sin(x)}(\cos(x))\,x - \ln(\sin(x))}{x^2}$$

$$\frac{dy}{dx} = y\left(\frac{\frac{\cos(x)}{\sin(x)}x - \ln(\sin(x))}{x^2}\right) = (\sin(x))^{1/x}\left(\frac{\frac{\cos(x)}{\sin(x)}x - \ln(\sin(x))}{x^2}\right)$$

(d) $x^2 = e^{x+y} + \sin(y^2)$

imp. diff. ↳ $$2x = e^{x+y}\left(1+\frac{dy}{dx}\right) + \cos(y^2)(2y)\frac{dy}{dx}$$

$$2x - e^{x+y} = e^{x+y}\frac{dy}{dx} + 2y\cos(y^2)\frac{dy}{dx}$$

$$2x - e^{x+y} = \left(e^{x+y} + 2y\cos(y^2)\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} + 2y\cos(y^2)}$$

[*5 marks*] **5.** Using L'Hôpital's Rule (if it is needed), evaluate the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{\log_2(x+1)}{x}$ $\quad \frac{0}{0}$

$\overset{L'H\hat{o}p}{=} \lim\limits_{x\to 0} \left(\dfrac{\frac{1}{(x+1)\ln(2)}}{1} \right)$

$= \dfrac{1}{\ln(2)}$

2.5 each

(b) $\lim\limits_{x\to\infty} x^2 \sin(1/x)$ $\quad (\infty)(0)$

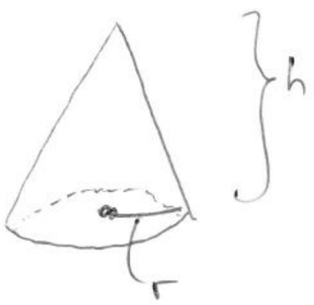
$= \lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x^2}$ $\quad \frac{0}{0}$

$\overset{L'H\hat{o}p}{=} \lim\limits_{x\to\infty} \dfrac{\cos(1/x)(-x^{-2})}{-2x^{-3}}$

$= \dfrac{x\cos(1/x)}{2}$ $\quad \dfrac{(\infty)(1)}{2}$

$= \infty$

[*7 marks*] **6.** A pile of sand forms a cone ($V = \frac{1}{3}\pi r^2 h$). Over time, rainfall wears away at the cone, causing the height of the cone to decrease, while maintaining a conical shape. None of the sand is blowing away, so the volume remains constant. The current height of the cone is $2m$ and its base radius is $3m$. If the height of the cone is decreasing at a rate of $0.01m/hr$, then how fast is the base radius increasing?



$\dfrac{dV}{dt} = 0$

Currently $h = 2$, $r = 3$

$\dfrac{dh}{dt} = -0.01$

Want: $\dfrac{dr}{dt}$

$V = \dfrac{\pi}{3} r^2 h$

imp. diff. w.r.t. 't'

$\dfrac{dV}{dt} = 0 = \dfrac{\pi}{3}\left(2r\dfrac{dr}{dt} h + r^2 \dfrac{dh}{dt} \right)$

$0 = \dfrac{\pi}{3}\left(2(3)\dfrac{dr}{dt}(2) + 9(-0.01) \right)$

$0 = 12\dfrac{dr}{dt} - 0.09$

$\rightarrow \dfrac{dr}{dt} = \dfrac{+0.09}{12} \approx 0.0075\, m/hr.$

The base radius is increasing at 0.0075m/hr.

[*13 marks*] **7.** Consider the following function on $[-3,3]$, along with its derivatives:

$$f(x)=\frac{x}{x^2+4} \qquad f'(x)=\frac{4-x^2}{(x^2+4)^2} \qquad f''(x)=\frac{2x(x^2-12)}{(x^2+4)^3}$$

(a) Find the roots/zeroes and vertical asymptotes of $f(x)$ (if there are any).

(b) Find the critical points of $f(x)$ and then use a sign chart to find the intervals where f is increasing/decreasing (if there are any).

2.5 each

(c) Find any local maximum and local minimum values (if there are any) and justify by the first or second derivative test. Determine the absolute maximum and minimum values on $[-3,3]$.

(d) Find the intervals of concavity (if there are any).

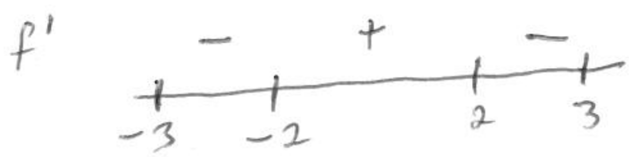
3 → (e) Use the information in (a)-(d) to sketch $f(x)$ on $[-3,3]$ labeling any important points. Draw your sketch neatly on the back page if you need more space.

a) $f(x)=0=\frac{x}{x^2+4} \rightarrow x=0$ is the only root.

$x^2+4>0$ (always), so no vert. asymptotes.

b) $0=f'(x)=\frac{4-x^2}{(x^2+4)^2} \rightarrow 0=4-x^2=(2+x)(2-x)$

$\underline{x=2}$ & $\underline{x=-2}$ are crit. pts.



inc. on $(-2,2)$ dec. on $(-3,-2)\cup(2,3)$

c) By the sign chart in (b), there is a local max @ $x=2$ and a local min @ $x=-2$.

$f(2)=\frac{1}{4}=0.25$ $f(3)=\frac{3}{13}\approx 0.231$

$f(-2)=-\frac{1}{4}=-0.25$ $f(-3)=-\frac{3}{13}\approx -0.231$

abs. max: 0.25 abs. min: −0.25

d) $f''=0=\frac{2x(x^2-12)}{(x^2+4)^3} \rightarrow 0=2x(x^2-12)$

$x=0$ or $x^2=12 \rightarrow x=\pm\sqrt{12}\approx\pm 3.46$ outside the domain.

concave up: $(-3,0)$

concave down: $(0,3)$

(infl. point: $x=0$.) – not asked for.

Extra space, if needed

root @ $x=0$

local max : $(2, \frac{1}{4})$ inc. $(-2,2)$

local min : $(-2,-\frac{1}{4})$ dec. $(-3,-2) \cup (2,3)$

concave up: $(-3,0)$

concave down $(0,3)$

e)