MAT3320 Assignment 2

Total: 10 marks. Due date: Oct. 7th;

1. (3 marks) Let $f(x) = x^3$, $2 < x < 6$. Find the Fourier-Legendre expansion.

   Solution: $P_n(x)$ are only defined on $-1 < x < 1$. So we need to make a linear transformation from (2,6) to (-1,1). Let $s = ax + b$. Then $-1 = a(2) + b, 1 = a(6) + b$. Then $a = 1/2, b = -2$, $s = \frac{1}{2}x - 2$. Let

   $$g(s) = f(x) = f(2s + 4) = (2s + 4)^3 = 8s^3 + 48s^2 + 96s + 64.$$

   Note that

   $$s^3 = \frac{2}{5}P_3(s) + \frac{3}{5}P_1(s); \quad s^2 = \frac{2}{3}P_2(s) + \frac{1}{3}P_0(s); s = P_1(s); 1 = P_0(s).$$

   We imply that

   $$g(s) = 8[\frac{2}{5}P_3(s) + \frac{3}{5}P_1(s)] + 48[\frac{2}{3}P_2(s) + \frac{1}{3}P_0(s)] + 96P_1(s) + 64P_0(s)$$

   $$= \frac{16}{5}P_3(s) + 32P_2(s) + \frac{504}{5}P_1(s) + 80P_0(s), \Rightarrow$$

   $$f(x) = \frac{16}{5}P_3(\frac{1}{2}x - 2) + 32P_2(\frac{1}{2}x - 2) + \frac{504}{5}P_1(\frac{1}{2}x - 2) + 80P_0(\frac{1}{2}x - 2).$$

2. (6 marks=1+1+2+2) Consider the following equation:

   $$2xy'' + y' + 3y = 0.$$

   (a) Show that $x_0 = 0$ is a regular singular point.

   (b) Write down the indicial equation and solve it to determine r_1 and r_2, $r_1 \geq r_2$.

   (c) Let $y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}$. Determine the recursive relation for $c_n(r)$, i.e., relation between $c_{n+1}(r)$ and $c_n(r)$.

   (d) Take $c_0(r) = 1$. Find two linearly independent solutions y_1 and y_2 which are valid for $x > 0$ near $x_0 = 0$.

   Solution: (a) We rewrite the equation as

   $$y'' + \frac{1}{2x}y' + \frac{3}{2x}y = 0.$$

   $a(x) = \frac{1}{2x}$ and $b(x) = \frac{3}{2x}$ are singular at $x_0 = 0$; $xa(x) = \frac{1}{2}$ and $x^2b(x) = \frac{3}{2}x$ are analytic at $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

(b) $p_0 = \frac{1}{2}$, $q_0 = 0$. Then we get the indicial equation: $r^2 - \frac{1}{2}r = 0$. Then $r_1 = \frac{1}{2}$, $r_2 = 0$. This is Case I.

(c) Substitute

$$y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}, y' = \sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1}, y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2}$$

into the differential equation to obtain

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n(r)x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+r)(2n+2r-1)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n(r)x^{n+r} = 0 \Rightarrow$$

$$r(2r-1)x^{r-1} + \sum_{n=1}^{\infty}[(n+r)(2n+2r-1)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n(r)x^{n+r} = 0 \Rightarrow$$

$$r(2r-1)x^{r-1} + \sum_{n=0}^{\infty}[(n+r+1)(2n+2r+1)c_{n+1}(r)x^{n+r} + \sum_{n=0}^{\infty} 3c_n(r)x^{n+r} = 0 \Rightarrow$$

$$r(2r-1)x^{r-1} + \sum_{n=0}^{\infty}[(n+r+1)(2n+2r+1)c_{n+1}(r) + 3c_n(r)]x^{n+r} = 0 \Rightarrow$$

$$(n+r+1)(2n+2r+1)c_{n+1}(r) + 3c_n(r), \Rightarrow$$

$$c_{n+1}(r) = \frac{-3c_n(r)}{(n+r+1)(2n+2r+1)}, n \geq 0.$$

(d) Taking $r = r_1 = \frac{1}{2}$. Then

$$c_{n+1}(\frac{1}{2}) = \frac{-3}{(n+\frac{1}{2}+1)(2n+1+1)}c_n(\frac{1}{2}) = \frac{-3}{(2n+3)(n+1)}c_n(\frac{1}{2}), \Rightarrow$$

$$c_n(\frac{1}{2}) = \frac{(-1)^n 6^n}{(2n+1)!}.$$

Now we take $r = r_2 = 0$. Then

$$c_n(0) = \frac{(-1)^n 3^n}{(1)(2)\cdots(n)(1)(3)\cdots(2n-1)} = \frac{(-1)^n 3^n}{n!(2n-1)!!} \quad or \quad \frac{(-1)^n 6^n}{(2n)!}.$$

Thus

$$y_1 = x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{(2n+1)!}x^n, \quad y_2 = \sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{(2n)!}x^n.$$

3. (1 mark) Find the general solution of

$$x^2y''(x) + xy'(x) + (2x^2 - 9)y(x) = 0, \quad x \neq 0.$$

Solution: Note that $\lambda^2 = 2$ and $\nu^2 = 9$, $\Rightarrow \lambda = \sqrt{2}$ and $\nu = 3$. Since ν is an integer,

$$y_1(x) = J_3(\sqrt{2}|x|), \quad y_2(x) = Y_3(\sqrt{2}|x|).$$