Part I: Multiple Choice Questions:

[*34 marks*]
- There is only one correct answer in each question.
- For Questions 1 to 7, answer all the multiple choice questions in the provided space including all the necessary steps and explanations. For each question, 1 mark for circling the right answer and 3 marks for the steps and explanation.
- For Questions 8 to 10, two marks for choosing the right answer in each question.

b

1. Suppose B and C ~~are~~ mutually exclusive events for which $P(B)$ and $P(C)$. Then $P(B|C)$ is
   (a) 1 (b) 0 (c) 0.5 (d) $P(B)$

$$P(B|C) = \frac{P(B\cap C)}{P(C)} = \frac{0}{P(C)} = 0$$

1 1 1

b

2. Suppose that 20% of all copies of a particular textbook fail a binding strength test. Let X denote the number among 15 randomly selected copies that fail the test. What is the probability that between ~~4~~ and 7 copies, inclusive, fail the test?
   (a)0.600 (b) 0.348 (c) 0.648 (d) 0.20

$p = 0.2 \quad n = 15 \quad P(x) = \binom{15}{x} 0.2^x \, 0.8^{15-x}$

$P(4 \le X \le 7) = F(7) - F(3) = B(7;15,.2) - B(3;15,0.2)$ 1

$= 0.996 - 0.648$ 1 ($\frac{1}{2}$ each)

$= 0.348$ 1

b

3. A random variable X has probability density function

$$f(x) = \begin{cases} \frac{5}{8\sqrt{2}}x^{3/2}, & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then $E(X^2)$ is: 1
(a)3.41 (b) 2.22 (c) 2.79 (d) 3.37

$$E(X^2) = \int_0^2 \frac{5}{8\sqrt{2}} x^{\frac{3}{2}} \cdot x^2 \, dx = \frac{5}{8\sqrt{2}} \int_0^2 x^{\frac{7}{2}} dx$$

$$= \frac{5}{8\sqrt{2}} \times \frac{1}{\frac{7}{2}+1} \left(x^{1+\frac{7}{2}}\right)_0^2 \quad 1$$

$$= \frac{5}{8\sqrt{2}} \times \frac{2}{9} \times 2^{\frac{9}{2}}$$

$$= \frac{5}{\cancel{8\sqrt{2}}} \times \frac{2}{9} \times \overset{2}{\cancel{16\sqrt{2}}}$$

$$= \frac{20}{9}$$

$$= 2.22$$

1

4. If the probabilities of having a male female child are 0.5, then the probability that a family's seven child is their second second daughter is:
(a)0.250 (b) 0.160 (c) 0.047 (d) 0.001

$k=2 \quad x=7 \quad p=0.5$ 1

c

$P(X=2) = \binom{7-1}{2-1} 0.5^{2}\, 0.5^{5} = 0.047$

1

d

5. A shipment of 20 digital voice recorders contains 5 that are defective. If 10 of them are randomly chosen without replacement for inspection, what is the probability that 2 of the 10 will be defective?
(a) 0.28 (b) 0.20 (c) 0.50 (d) 0.35

$N=20 \quad a=5 \quad n=10 \quad x=2$ 1

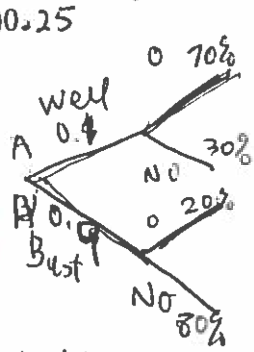
$$P(X=2) = \frac{\binom{N-a}{n-x}\binom{a}{x}}{\binom{N}{n}} = \frac{\binom{20-5}{10-2}\binom{5}{2}}{\binom{20}{10}} = \frac{\binom{15}{8}\binom{5}{2}}{\binom{20}{10}}$$

1 1

$\doteq 0.35$

a

6. Ashley is hoping to land a summer internship. If her interview goes well, she has a 70% chance of getting an offer. If the interview is a bust, her chance drops to 20%. If under stress the probability of the interview going well is only 0.10, what is the probability that Ashley gets the internship?
(a) 0.25 (b) 0.07 (c) 0.18 (d) 0.09



1

B : offer A : well / not

1 $P(B) = P(B|A)\cdot P(A) + P(B|A')\cdot P(A')$

1 $= 0.1 \times 0.7 + 0.9 \times 0.2$

$= 0.07 + 0.18 = 0.25$

7. If a typist averages one misspelling in every 3250 words, what is the chance that a 6500-word report has at least 3 errors?

(a) 0.677
(b) 0.500
(c) 0.323
(d) 0.143
(e) not possible to calculate using the given information.

c

[1.5 if using binomial distribution]

$p = \frac{1}{3250}, \quad n = 6500$

$\lambda = np = 6500 \times \frac{1}{3250} = 2$ 1

$P(x) = \frac{\lambda^{x} e^{-\lambda}}{x!}$ 1

$P(X \geq 3) = 1 - P(X<3) = 1 - P(X \leq 2)$

$= 1 - 0.677$

$= 0.323$ 1

8. Which one of the following statements does not apply to the of Bernoulli trials?

2

a

(a) A sequence of Bernoulli trials lead to a hypergeometric distribution.
(b) In a sequence of Bernoulli trials, the probability of success in one trial is not determined by the probability of success of the other trials.
(c) There are exact two possible outcomes in a Bernoulli trial.
(d) A sequence of Bernoulli trials are independent repeated trials.

2

9. Suppose that X is a discrete random variable that follows a binomial distribution with n trials and a constant probability of success, p. When n goes to infinity, which one of the following distributions will be a good approximation of the distribution of X?

c

(a) Hypergeometric
(b) Poisson
(c) Normal
(d) Bernoulli

2

10. Which one of the following statements is incorrect?

d

(a) The mean and median of a normal distribution are the same.
(b) Suppose that X is a continuous random variable, then $P(X = 1.96)$ is 0.
(c) If two events are independent, then they may not be mutually exclusive.
(d) If correlation of X and Y is 1, then the correlation of X^2 and Y is also 1.

Part II – Written Questions

[*6 marks*] 1. The probability distribution of a discrete random variable X is

x	0	1	4
$p(x)$	0.125	0.25	0.625

(a) Find $P(X < 3)$.
(b) Find $Var(\sqrt{X})$

1 1

(a) $P(X<3) = P(X=0) + P(X=1) = 0.125 + 0.25 = 0.375$

(b) $Var(\sqrt{X}) = E(X) - E(\sqrt{X})^2$ 1

$E(X) = p(0) \times 0 + P(1) \times 1 + P(4) \times 4$ ✓
$= 0.125 \times 0 + 0.25 \times 1 + 0.625 \times 4$
$= 0.25 + 2.5$ 1.5
$= 2.75$

$E(\sqrt{X}) = P(0) \times \sqrt{0} + P(1) \times \sqrt{1} + P(4) \times \sqrt{4}$ ✓
$= 0 + 0.25 \times 1 + 0.625 \times 2$
$= 1.5$ 1.5

$Var(\sqrt{X}) = 2.75 - 1.5^2 = 0.5$

[*10 marks*] 2. MENSA is an international society devoted to intellectual pursuits. Any person who has an IQ that is at least 133 can join MENSA. Assume that IQs are normally distributed with mean 100 and standard deviation 16.

(a) What percent of the general population would be in MENSA?

(b) What is the 90^{th} percentile of the IQ score?

Solution:

(a) $X \sim N(100, 16^2)$

$$P(X \geq 133) = P\left(\frac{X-100}{16} \geq \frac{133-100}{16}\right) \; 1 \; = P(Z \geq 2.0625) \; 1$$

$$= 1 - P(Z < 2.0625) \quad 1$$

$$= 1 - F(2.06)$$

$$= 1 - 0.9803$$

$$= 0.0197 \quad 1$$

$\therefore$ percent of the general population would be in MENSA is 1.97%.

(b) $P(X \leq \eta(0.90)) = 0.9$ 1, $\eta(0.90) = ?$

$$P\left(Z \leq \frac{\eta - 100}{16}\right) = 0.9 \quad 1$$

From Table, we have

$$\frac{\eta - 100}{16} = 1.28 \quad 2$$

$$\eta = 1.28 \times 16 + 100 = 120.48. \quad 1$$

$\therefore$ The 90^{th} percentile of the IQ scores 120.48

[*10 marks*] 3. Suppose that Ω is a nonempty sample space and that A and B are events of Ω with $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$.

(a) Are A and B independent. Justify your answer.

(b) Are A and B mutually exclusive. Justify your answer.

(c) State the Bayes Rule for events A and B

(a) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ 1

$= 0.5 + 0.6 - 0.8$ 1

$= 0.3$.

$P(A) \cdot P(B) = 0.5 \times 0.6 = 0.3$ 1

$\because P(A \cap B) = P(A) \cdot P(B)$ 2

$\therefore$ A and B are independent 1

(b) $\because P(A \cap B) \neq 0$ 1 $\therefore$ A and B are not mutually exclusive 1

(c) $$P(A|B) = \frac{P(A \cap B)}{P(B|A) + P(B|A')}$$

2

[*10 marks*] 4. Suppose Y is a continuous random variable with probability density function (pdf)

$$f_Y(y) = \begin{cases} ky, & 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value k;

(b) Find the cumulative distribution function (cdf) of Y. Remember to consider the different cases (there are three);

(a) $\int_0^2 f_Y(y)\,dy = 1$ 1 $\quad \int_0^2 ky\,dy = 1 \quad k = \dfrac{1}{\int_0^2 y\,dy}$ 1

$$k = \frac{1}{\frac{1}{2}y^2\big|_0^2} = \frac{2}{2^2} = \frac{1}{2}. \quad 1$$

(b) $y \le 0 \qquad 0 < y < 2 \qquad y \geq 2$

$$0 \qquad\qquad 2$$

* $y \le 0$,
$$F(y) = \int_{-\infty}^{y} f(t)\,dt = \int_{-\infty}^{y} f(t)\,dt$$
$$= \int_{-\infty}^{y} 0\,dt = 0. \quad 1$$

* $0 < y < 2$
$$F(y) = \int_{-\infty}^{y} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^y f(t)\,dt$$
$$= \int_{-\infty}^{0} 0\,dt + \int_0^y \tfrac{1}{2}t\,dt$$
$$= \tfrac{1}{2} \cdot \tfrac{1}{2}t^2\big|_0^y$$
$$= \tfrac{1}{4}y^2$$

* $y \geq 2$
$$F(y) = \int_{-\infty}^{y} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^2 f(t)\,dt + \int_2^y f(t)\,dt$$
$$= \int_{-\infty}^{0} 0\,dt + \int_0^2 \tfrac{1}{2}t\,dt + \int_2^y 0\,dt$$
$$= \tfrac{1}{2} \times \tfrac{1}{2}t^2\big|_0^2 = \tfrac{1}{4} \times 2^2 = 1$$

$$\therefore\ F(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{4}y^2 & 0 < y < 2 \\ 1 & y \geq 2 \end{cases} \quad 1$$

[20 marks] 5. Let X and Y be two discrete random variables and their joint probability mass function (pmf), $P(x, y)$, is given in the following table.

$P(x,y)$	$x=0$	$x=1$	$x=2$	$x=5$	P(y)
$y=0$	0.04	0.06	0.05	0.03	0.15
$y=5$	0.07	0.10	0.20	0.10	0.47
$y=10$	0.02	0.03	0.15	0.15	0.35
P(x)	0.13	0.19	0.40	0.28	

(a) Find the marginal probability mass function of X and Y;

(b) Find the conditional probability mass function of Y, given $X = 5$;

(c) Find the probability that $X + Y \leq 2.5$;

(d) Find the covariance between X and Y, $Cov(X, Y)$, given that $E(X) = 2.39$ and $E(Y) = 5.85$.

(e) Are X and Y correlated? Justify your answer.

a).

x	0	1	2	5
$P(x)$	0.13	0.19	0.40	0.28

2

y	0	5	10
$P(y)$	0.15	0.47	0.35

2

b) $P(Y \mid X=5) = \dfrac{P(y,5)}{P(5)}$ 2

$P(0,5) = \dfrac{0.03}{0.28}$

$P(5,5) = \dfrac{0.10}{0.28}$

$P(10,5) = \dfrac{0.15}{0.28}$

c) $P(X+Y \leq 2.5) = P(0,0) + P(1,0) + P(2,0)$

$= 0.04 + 0.06 + 0.05 = 0.15$

d) $E(XY)$

$= \sum\sum xy\, P(x,y)$ 2

$= (1\times5\times0.1 + 2\times5\times0.20 + 5\times5\times0.10 + 1\times10\times0.03$
$+ 10\times2\times0.15 + 5\times10\times0.15)$ 1

$= (0.5 + 2 + 2.5 + 0.3 + 3 + 7.5)$ 1
$= 15.8$

$Cov(X,Y) = E(XY) - E(X)E(Y)$ 1

$= 15.8 - 2.39 \times 5.85$ 1
$= 1.8185$

e) Yes 1 $\because Cov(X,Y) \neq 0$ 1

[*5 marks*] 6. Let X be a random variable and a a constant. Prove that $Var(aX) = a^2 Var(X)$.

Proof: $Var(aX)$

$= E[(aX)^2] - [E(aX)]^2$ 1

$= E[a^2X^2] - [aE(X)]^2$ 2

$= a^2 E(X^2) - a^2 E(X)^2$ 3

$= a^2 [E(X^2) - E(X)^2]$ 4

$= a^2 Var(X)$. 1

[*5 marks*] 7. Show that $P(A) = P(A \cap B) + P(A \cap B')$. (Hint: Use a Venn diagram.)

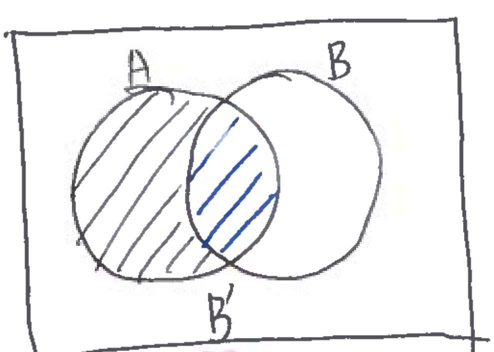
$A = (A \cap B') \cup (A \cap B)$ 1

$\because$ $A \cap B'$ and $A \cap B$ are disjoint 1

$\therefore$ $P[(A \cap B') \cup (A \cap B)]$ 1

$= P[(A \cap B')] + P[(A \cap B)]$

By A3.



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