MAT2377C - Assignment 2

Q1. (6 points) The sample space of a random experiment is $\{a, b, c, d, e, f\}$ and each outcome is equally likely. A random variable is defined as follows

outcome	a	b	c	d	e	f
x	0	0	1.5	1.5	2	3

Determine the probability mass function of X. Determine the following probabilities:

(a) $P(X = 1.5)$ (b) $P(0.5 < X < 2.7)$ (c) $P(X > 3)$

(d) $P(0 \le X < 2)$ (e) $P(X = 0 \text{ or } X = 2)$

Solution to Q1:

Probability mass function is

$$P(X = 0) = P(\{a, b\}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6};\ P(X = 1.5) = \frac{2}{6},\ P(X = 2) = \frac{1}{6},\ P(X = 3) = \frac{1}{6}.$$

(a) $P(X = 1.5) = \frac{2}{6}$

(b) $P(0.5 < X < 2.7) = P(X = 1.5) + P(X = 2) = \frac{3}{6}$

(c) $P(X > 3) = 0$

(d) $P(0 \le X < 2) = P(X = 0) + P(X = 1.5) = \frac{4}{6}$

(e) $P(X = 0 \text{ or } X = 2) = \frac{3}{6}$

Marking scheme for Q1:

Completely correct p.m.f. - 1 point, correct answer for each part - 1 point. Total - 6 points.

Q2. (2 points) Determine the mean and the variance in Question Q1

Solution to Q2:

$\mathrm{E}(X) = 1.33$, $\mathrm{Var}(X) \approx 1.15$

Marking scheme for Q2:

1 point for the mean and the variance. Total - 2 points.

Q3. We say that X has *uniform distribution* on a set of values $\{x_1, \ldots, x_k\}$ if

$$P(X = x_i) = \frac{1}{k}, \qquad i = 1, \ldots, k.$$

The thickness measurements of a coating process are *uniformly distributed* with values 0.15, 0.16, 0.17, 0.18, 0.19. Determine the mean and variance.

Solution to Q3:

We have $P(X = 0.15) = \cdots = P(X = 0.19) = 1/5$. Mean: 0.17; Variance:

$$\frac{1}{5}\left(0.01^2 + 0.02^2 + 0.02^2 + 0.01^2\right)$$

Marking scheme for Q3:

This question will not be marked

Q4. (3 points) Samples of rejuvenated mitochondria are mutated in 1% cases. Suppose 15 samples are studied and they can be considered to be independent for mutation. Determine the following probabilities:

(a) No samples are mutated

(b) At most one sample is mutated

(c) More than half the samples are mutated

Solution to Q4:

Let X be the number of mutated samples; then X has binomial distribution with $n = 15$ and $p = 0.01$ (success=mutation)

(a) To compute $P(X = 0) = \binom{15}{0} 0.01^0 0.99^{15} = 0.86$

(b) at most one sample mutated $= P(X = 0) + P(X = 1) = 0.86 + \binom{15}{1} 0.01^1 0.99^{14} = 0.99$

(c) More than half the samples are mutated:

$$P(X = 8) + \cdots + P(X = 15) \approx 0$$

Marking scheme for Q4:

Correct answer for each part - 1 point. Total - 3 points.

Q5. A statistical process control (6 points). Samples of 20 parts from a metal punching process are selected every hour. Typically, 1% of the parts require rework. Let X denote the number of parts in the sample that require rework. A process problem is suspected if X exceeds its mean by more than three standard deviations.

(a) What is the probability that there is a process problem?

(b) If rework percentage increases to 4%, what is the probability that X exceeds 1?

Solution to Q5:

(a) We have $X \sim \mathcal{B}(n, p)$, $n = 20$, $p = 0.01$ (success = a part requires rework). $\mathrm{E}(X) = np = 0.2$, $\mathrm{Var}(X) = np(1 - p) = 0.2 \times 0.99 = 0.198$, $\mathrm{SD}(X) \approx 0.44$. To compute

$$\begin{aligned} P(X > 0.2 + 3 \times 0.44) &= P(X > 1.535) = P(X \geq 2) = 1 - P(X < 2) = 1 - (P(X = 0) + P(X = 1)) \\ &= 1 - \binom{20}{0} 0.01^0 0.99^{20} - \binom{20}{1} 0.01^1 0.99^{19} \approx 0.017 \end{aligned}$$

(b) We have $X \sim \mathcal{B}(n, p)$, $n = 20$, $p = 0.04$. To compute

$$\begin{aligned} P(X > 1) &= P(X \geq 2) = 1 - P(X < 2) = 1 - (P(X = 0) + P(X = 1)) \\ &= 1 - \binom{20}{0} 0.04^0 0.96^{20} - \binom{20}{1} 0.04^1 0.96^{19} \approx 0.19. \end{aligned}$$

Marking scheme for Q5:

For part a) - 3 points: 1 point for correct $\mathrm{E}(X)$ and $\mathrm{SD}(X)$, 1 point for correct formula to compute, i.e. $P(X > 0.2 + 3 \times 0.44)$, 1 point for the correct answer. 1 point for part b). 2 points for part c): 1 point for the correct identification of Y , 1 point for the correct answer. Total - 6 points.

Q6. (4 points) In a clinical study, volunteers are tested for a gene that has been found to increase the risk for a disease. The probability that the person carries a gene is 0.1

(a) What is the probability that 4 or more people will have to be tested in order to detect one person with the gene?

(b) How many people are expected to be tested in order to detect one person with the gene?

(c) How many people are expected to be tested before 2 with a gene are detected?

Solution to Q6:

(a) If X is the number of steps before the 1st success, then X has geometric distribution with $p = 0.1$ (success = gene detection). To compute

$$P(X \geq 4) = \sum_{k=4}^{\infty} (1-p)^{k-1} p = p \times \frac{(1-p)^3}{1-(1-p)} = (1-p)^3 = 0.729.$$

(b) $\mathrm{E}(X) = 1/p = 10$.

(c) $\mathrm{E}(X) = 20$.

Marking scheme for Q6:

Correct answer for each part - 1 point. Total - 4 points.

Q7. The number of failures of a testing instrument from contamination particles on the product is a Poisson random variable with a mean of 0.02 failure per hour.

(a) What is the probability that the instrument does not fail in an 8-hour shift?

(b) What is the probability of at least one failure in a 24-hour day?

Solution to Q7:

(a) The failure rate per 8 hours is $8 \times 0.02 = 0.16$. If X is Poisson random variable with $\lambda = 0.16$, we have to compute $P(X = 0) = \exp(-0.16)$.

(b) Now, X is Poisson with $\lambda = 24 \times 0.02 = 0.48$. To compute $P(X \geq 1) = 1 - P(X = 0) = 1 - \exp(-0.48)$.

Marking scheme for Q7:

This question will not be marked