MAT1341D - Test 2 - Version 1

9-February, 14:30-15:50, 2015 Duration: 80 minutes

Instructor: X. Hua

Question	Response
1	
2	
3	
4	
5	
6	
[Bonus] 7	
Total	

Family Name:______________________

First Name:_______________________

Student Number: __________________

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY

1. This is a closed book exam, and no notes of any kind are allowed. **Do not use your own scrap paper! Use the backs of pages for rough work**.
2. The use of calculators, cell phones, pagers or any text storage or communication device is not permitted.
3. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. Please record your answers in the space provided above.
4. Questions 4 – 6 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.** Question 7 is a bonus question and is worth 3 points.
5. Good luck! Bonne chance!

1. Let $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x\,y\,w = 0\}$. Then,

A. $(0, 0, 0, 0) \in U$ but U is not closed under multiplication by scalars

B. U is closed under addition and U is closed under multiplication by scalars

C. U is closed under addition but U is not closed under multiplication by scalars

D. U is not closed under addition but U is closed under multiplication by scalars

E. $(0, 0, 0, 0) \notin U$ but U is closed under addition

F. None of the other statements is true.

Solution: D.

For example, $(0, 1, 1, 1), (1, 0, 1, 3) \in U$, but $(0, 1, 1, 1) + (1, 0, 1, 3) = (1, 1, 2, 4) \notin U$, since $1(1)(4) \neq 0$.

2. Which of the following are subspaces of $\mathbb{R}^3$?

$U = \{(x,\ y,\ z) \mid 2x - y + 3z = 0\}$

$V = \{(x,\ y,\ z) \mid xy = 0\}$

$W = \{(x,\ y,\ z) \mid 2x = 5z\}$

$X = \{(x,\ y,\ z) \mid x = y + 3 = 7z\}$

A. U and V

B. U, W and X

C. W and X

D. U and W

E. V and X

F. V and W

Solution: D.

$U = span\{(1, 2, 0), (0, 3, 1)\}$, $W = span\{(0, 1, 0), (5/2, 0, 1)\}$

V is not closed under addition, X does not satisfy any of the Three Subspace Test conditions.

3. Which two of the following are subspaces of $\mathbf{F}[\mathbb{R}] = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$?

$S = \{f \in \mathbf{F}[\mathbb{R}] \mid f(0) = 1\}$

$T = \{f \in \mathbf{F}[\mathbb{R}] \mid f(1) = 0\}$

$U = \{f \in \mathbf{F}[\mathbb{R}] \mid f(0)f(1) = 0\}$

$V = \{f \in \mathbf{F}[\mathbb{R}] \mid f(x) = f(-x), \quad \forall x \in \mathbb{R}\}$

A. S and V.

B. T and U.

C. S and T.

D. T and V.

E. S and U.

F. V and U.

Solution: D.

S does not satisfy any of the Three Subspace Test conditions.

U is not closed under addition: for example, $f(x) = x \in U$, $g(x) = 1 - x \in U$, but $f(x) + g(x) = 1 \notin U$.

4. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + 3z = 0\}$.

a) Explain very briefly why W is a subspace of $\mathbb{R}^3$.

b) Find a spanning set for W.

c) Give a complete geometric description of W. *(Remember that you must justify your answers.)*

Solution: a) It is a plane through the origin; every plane through the origin is a subspace of $\mathbb{R}^3$. It's ok if you use Subspace Test.

Marking scheme: (1.5 points: It is a plane[.5] through the origin[.5]; every plane through the origin is a subspace of $\mathbb{R}^3$[.5])

b) For any $(x, y, z) \in W$, $y = x + 3z$. Thus $(x, y, z) = (x, x + 3z, z) = (x, x, 0) + (0, 3z, z) = x(1, 1, 0) + z(0, 3, 1)$.

$W = span\{(1, 1, 0), (0, 3, 1)\}$, or $W = span\{(1, 1, 0), (-3, 0, 1)\}$, or...

Marking scheme: [3=1 (knowing what 'spanning set' means + .5 some correct work + 1 vector(s) in W + .5 all correct]

c) It is a plane through the origin, with normal vector $n = (1, -1, 3)$.

Marking scheme: [1.5= .5 'plane' + .5 through (correct) 'point')+ .5 with (correct) normal 'n'.]

5. Let $\mathbf{M}_{22}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in \mathbf{M}_{22} \mid a, b \in \mathbb{R} \right\}.$$

a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees that U is a subspace.

(For (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{22}$.)

b) Find a spanning set for U.

c) Give a matrix $A \in \mathbf{M}_{22}$ such that $A \notin U$. *(Remember that you must justify your answers.)*

Solution: a) Let $\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix} \in U$.

$$\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & 0 \end{bmatrix} \in U.$$

Thus U is closed under addition.

"or"

$U = span\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$. Thus U is a subspace.

Marking scheme: 2 points: = 1 if they simply say it a span, + 1 for justification of this statement; if check that U is closed under addition, the justification must be totally correct to get 2; examples to prove it is closed get (maximum) 0.5

b) For any $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in U$,

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus

$$U = span\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}.$$

Marking scheme: [3=1 (knowing what 'spanning set' means + .5 (some correct work) + 1 (elements actually in U) + .5 all correct]

c) For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin U$.

Marking scheme: 1 point

6. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

•If you say the statement may be false, you

must give an explicit example - with numbers, or functions, as is appropriate!

•If you say the statement is always true, you must give a clear explanation.

a) $X = \{f \in \mathbf{F}(\mathbb{R}) \mid f(x) \leq 0 \quad \text{for all } x \in \mathbb{R}\}$ is a subspace of $\mathbf{F}(\mathbb{R})$

Solution: False.

X is not closed under scalar multiplication. For example, $f(x) = -1 \in X$, but $-2f(x) = -2(-1) = 2 \notin X$.

Marking scheme: [1.5= .5 (correct) + 1 justification(.5 is possible here)]

ANSWER

b) If u and v are vectors in $\mathbb{R}^2$ and U is a subspace of $\mathbb{R}^2$ with $u - v \in U$, then both u and v belong to U.

Solution: False.

Let $U = span\{(1,1)\}$. Then $(4,3) - (3,2) = (1,1) \in U$. However, $(4,3)$, $(3,2) \notin U$.

Marking scheme: [1.5= .5 (correct) + 1 justification(.5 is possible here)]

ANSWER

c) $\left\{ \begin{bmatrix} a & a \\ b & c \end{bmatrix} \in \mathbf{M}_{22} \mid a, b, c \in \mathbb{R} \right\}$ is a subspace of $\mathbf{M}_{22}$.

Solution: True.

Since $\begin{bmatrix} a & a \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$\left\{ \begin{bmatrix} a & a \\ b & c \end{bmatrix} \in \mathbf{M}_{22} \mid a, b, c \in \mathbb{R} \right\} = span\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$. Any span is a subspace.

Marking scheme: [1.5= .5 (correct) + 1 justification(.5 is possible here)]

ANSWER

d) If u, v belong to a vector space V then $span\{u, v\} = span\{u, u + v\}$.

Solution: True.

"$\Rightarrow$" For any $w \in span\{u, v\}$, there exist $c, d \in \mathbb{R}$ such that

$w = cu + dv = (c - d)u + d(u + v) \in span\{u, u + v\}$.

$\Rightarrow span\{u, v\} \subset span\{u, u + v\}$;

"$\Leftarrow$" For any $w \in span\{u, u + v\}$, there exist $c, d \in \mathbb{R}$ such that

$w = cu + d(u + v) = (c + d)u + dv \in span\{u, v\}$, $\Rightarrow span\{u, u + v\} \subset span\{u, v\}$;

Thus $span\{u, v\} = span\{u, u + v\}$.

Marking scheme: [1.5= .5 (correct) + 1 justification(.5 is possible here)]

ANSWER

7. [Bonus] Give the set $U = \{(x-2, x) \mid x \in \mathbb{R}\}$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x' + 2,\ y + y') \quad \text{(vector addition)}$$

and

$$k \odot (x,\ y) = (kx + 2k - 2,\ ky) \quad \text{(multiplication by scalars).}$$

a) Prove that U is closed under the operation of vector addition defined above.

b) Show that U *has* a zero vector. (i.e. find it and show it works.).

Solution:

Proof of (a): Let $u = (x, y) \in U$, $v = (x', y') \in U$, then $x = y - 2$ and $x' = y' - 2$.

$$u \oplus v = (x,\ y) \oplus (x',\ y') = (x + x' + 2,\ y + y') = (y - 2 + y' - 2 + 2,\ y + y')$$
$$= (y + y' - 2,\ y + y') \in U.$$

Marking scheme: 1.5 points- must be well-argued and clear to get 1.5

Solution: (b): Let $\vec{0} = (x', y')$, then for any $(x, y) \in U$,

$$(x,\ y) \oplus (x',\ y') = (x,\ y), \Rightarrow$$
$$(x + x' + 2,\ y + y') = (x,\ y), \Rightarrow$$
$$x + x' + 2 = x, y + y' = y, \Rightarrow x' = -2, y' = 0.$$

Thus $\vec{0} = (-2, 0)$.

Marking scheme: 1.5 points= .5 correct zero vector + 1 -justified; must be well-argued and clear to get 1.5