Concordia University

Course	Number	Sections
ENGR	233	P, Q

Examination	Date	Time	Total Marks	Pages
Final	April 2008	3 hours	100	2

Course Coordinator	Instructors
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Special Instructions: use of calculators and outside materials is NOT permitted.

Each problem is worth 10 marks unless stated otherwise.

Problem 1. For the vector field

$$\vec{F}(x,y,z) = x^2y\mathbf{i} + xy^2\mathbf{j} + 2xyz\mathbf{k}$$

compute –**if possible**– the following quantities. If it is not possible **explain why not**.

(a) div(curl $\vec{F}(x,y,z)$) , (b) curl($\underbrace{\text{div}\,\vec{F}(x,y,z)}_{\text{scalar}}$) , (c) grad(div $\vec{F}(x,y,z)$) , (d) div(grad $\vec{F}(x,y,z)$)

a) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xy^2 & 2xyz \end{vmatrix} = \langle 2xz, -2yz, y^2 - x^2 \rangle$

$\text{div}(\text{curl}F) = 2z - 2z - 0 = 0$

b) $\text{div}F = 2xy + 2xy + 2xy = 6xy$ $\quad \nabla\text{div}F = \langle 6y, 6x, 0 \rangle$

Problem 2. Find the equation of the tangent plane of the surface defined by

$$z^3 - xyz = 1$$

at the point $(4, \frac{1}{2}, -1)$.

$n = \langle -yz, -xz, 3z^2 - xy \rangle$ plug in $(4, \frac{1}{2}, -1)$ $\vec{n} = \langle \frac{1}{2}, 4, 1 \rangle$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\frac{1}{2}(x - 4) + 4(y - \frac{1}{2}) + (z + 1) = 0$

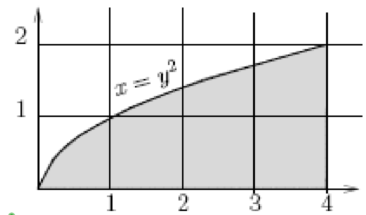
$\frac{1}{2}x - 2 + 4y - 2 + z + 1 = 0$

$\frac{1}{2}x + 4y + z = 3$

Problem 3.
Evaluate the following integral by reversing the order of integration

$$\int_0^2 \int_{y^2}^4 e^{\sqrt{x^3}}\,dx\,dy$$

[Hint: the following substitution may be of help: $u = x^{\frac{3}{2}}$]



$$\int_0^4 \int_0^{\sqrt{x}} e^{x^{3/2}}\,dy\,dx = \int_0^4 \sqrt{x}\, e^{x^{3/2}}\,dx$$

$u = x^{3/2} \longrightarrow x=0\ u=0,\ x=4\ u=8$

$dx = \dfrac{du}{\frac{3}{2}x^{1/2}} = \dfrac{2du}{3\sqrt{x}}$

$$= \int_0^8 \cancel{\sqrt{x}} \cdot e^u \cdot \frac{2du}{3\cancel{\sqrt{x}}} = \frac{2}{3}e^u\Big|_0^8 = \frac{2}{3}\left(e^8 - 1\right)$$

Problem 4. Find the rate of change at the point $(2, 1, 3)$ of the following function $f(x, y, z) = \frac{xy}{z^2}$ along the directions given by unit vectors parallel to
(a) $\mathbf{i}$; (b) $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

unit vectors

a) $\langle 1,0,0\rangle$ (b) $\dfrac{\langle 1,2,-1\rangle}{\sqrt{1+4+1}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right\rangle$

$$\nabla f = \left\langle \frac{y}{z^2}, \frac{x}{z^2}, \frac{-2xy}{z^3}\right\rangle$$

$$\nabla f(2 1 3) = \left\langle \frac{1}{9}, \frac{2}{9}, \frac{-4}{27}\right\rangle$$

directional derivatives:

a) $\left\langle \frac{1}{9}, \frac{2}{9}, \frac{-4}{27}\right\rangle \cdot \langle 1,0,0\rangle = \frac{1}{9}$

b) $\left\langle \frac{1}{9}, \frac{2}{9}, \frac{-4}{27}\right\rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right\rangle =$

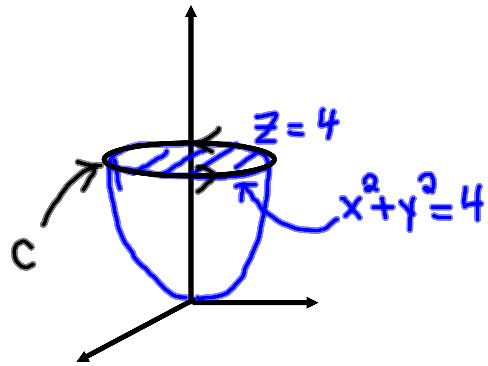
$$\frac{1}{9\sqrt{6}} + \frac{4}{9\sqrt{6}} + \frac{4}{27\sqrt{6}} = \frac{19}{27\sqrt{6}}$$

Problem 5. Using **Stokes' theorem**, compute the flux of the curl of the vector field

$$\vec{F}(x,y,z) = 6yz\mathbf{i} - 24x\mathbf{j} + yze^{x^2+\arctan(z)}\mathbf{k}$$

across the surface S (oriented upwards) of the paraboloid $z = y^2 + x^2$, $z \le 4$, with boundary the circle $z = 4, x^2 + y^2 = 4$.

$$\oint_C F \cdot dr = \iint_S (\text{curl} F) \cdot n\,dS$$

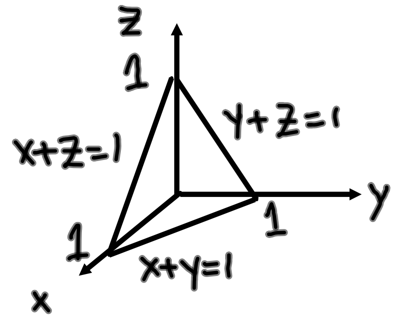


$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle \quad t: 0 \to 2\pi$$

$$dr = \langle -2\sin t^{*}, 2\cos t^{**}, 0 \rangle$$

$$\iint_S \text{curl} F \cdot n\,dS = \int_0^{2\pi} -96\sin^2 dt - 96\cos^2 dt = -96(2\pi) = -192\pi$$

Problem 6. Find the mass $M = \iiint_{\mathcal{R}} \rho(x,y,z)dV$ of the solid in the first octant (namely $x \geq 0, y \geq 0, z \geq 0$) bounded by the coordinate planes and the graph of $x+y+z=1$ if the density is given by $\rho = x + 2y$.



$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} (x+2y)\, dx\, \underset{\overset{\uparrow}{y,z}}{d}y\, \underset{\overset{\uparrow}{z}}{d}z$$

$$= \int_0^1 \int_0^{1-z} \left. \frac{x^2}{2} + 2xy \right|_0^{1-y-z} dy\, dz$$

$$= \int_0^1 \int_0^{1-z} \frac{(1-y-z)^2}{2} + 2y(1-y-z)\ dy\, dz \ \ldots$$

Problem 7.

$\curvearrowright$ curl F = 0

Evaluate the work done by the **conservative** force

$$\vec{F}(x, y) = ye^{xy}\mathbf{i} + (xe^{xy} + 2y)\mathbf{j}$$

along any path that joins the starting point $(0, 0)$ and ending point $(1, 2)$. **You must use the potential function.**

$$\phi_x = \int ye^{xy}dx = e^{xy} + g(y) \qquad \phi_y = \int(xe^{xy} + 2y)dy = e^{xy} + y^2 + h(x)$$

$$\phi = e^{xy} + y^2 \qquad \text{work} = \phi(1,2) - \phi(0,0) = e^2 + 4 - 1 = e^2 + 3$$

Problem 8. Find the curvature $\varkappa(t)$ and the components of the acceleration $a_N(t), a_T(t)$ for the curve described by

$$r = \langle t+1, t^2-t, e^{-t} \rangle \quad \mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-t)\mathbf{j} + e^{-t}\mathbf{k}$$

$$r' = \langle 1, 2t-1, -e^{-t} \rangle$$

$$r'' = \langle 0, 2, e^{-t} \rangle$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t-1 & -e^{-t} \\ 0 & 2 & e^{-t} \end{vmatrix} = \langle 2te^{-t} - e^{-t} + 2e^{-t}, e^{-t}, 2 \rangle$$

$$= \langle e^{-t}(2t+1), e^{-t}, 2 \rangle$$

$$\|r' \times r''\| = \sqrt{e^{-2t}(2t+1)^2 + e^{-2t} + 4}$$

$$K = \frac{\sqrt{e^{-2t}(2t+1)^2 + e^{-2t} + 4}}{\left(1 + (2t-1)^2 + e^{-2t}\right)^{3/2}}$$

$$a_T = \frac{\|r' \cdot r''\|}{\|r'\|} = \frac{\|\langle 0, 4t-2, -e^{-2t} \rangle\|}{\sqrt{1 + (2t-1)^2 + e^{-2t}}}$$

$$= \sqrt{\frac{(4t-2)^2 + e^{-4t}}{1 + (2t-1)^2 + e^{-2t}}}$$

$$a_N = \frac{\sqrt{e^{-2t}(2t+1)^2 + e^{-2t} + 4}}{\sqrt{\left(1 + (2t-1)^2 + e^{-2t}\right)}}$$

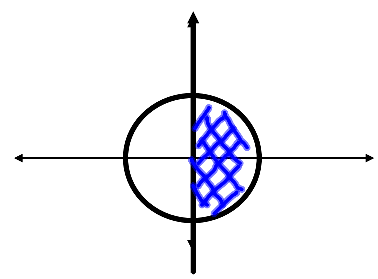
Problem 9. Use **Green's theorem** to compute the line-integral

$$\oint_C y^2\,\mathrm{d}x + x\,\mathrm{d}y = \iint_R (1-2y)\,dA$$

where C is the boundary of the region determined by the graphs of $x = 0$, $x^2 + y^2 = 4$ and with $x \geq 0$.

in polar coordinates:

$$\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^2 (1-2r\sin\theta)\,r\,dr\,d\theta$$



$$= \int_{-\pi/2}^{\pi/2} \frac{r^2}{2} - \frac{2r^3}{3}\sin\theta \Big|_0^2 \,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2 - \frac{16}{3}\sin\theta\right) d\theta = 2\theta + \frac{16}{3}\cos\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \pi - (-\pi) = 2\pi$$

Problem 10. Use the **Divergence Theorem** to evaluate the outward flux $\iint_S \vec{F}\cdot\vec{n}\,dS$ of the given vector field across the surface specified

$$\vec{F}(x,y,z) = x^3\mathbf{i} + (y^3+xz)\mathbf{j} + (z^3+z^2)\mathbf{k}$$

$$x^2+y^2+z^2=a^2, \quad a>0. \leftarrow \text{sphere of radius } a$$

$$\iint_S F\cdot n\,ds = \iiint_V \operatorname{div}F\,dV = \iiint_V (3x^2+3y^2+3z^2)\,dV$$

spherical coordinates:

$$3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 3\int_0^{2\pi}\int_0^{\pi}\frac{a^5}{5}\sin\phi\,d\phi\,d\theta = \frac{3a^5}{5}\int_0^{2\pi} -2\,d\theta$$

$$= \frac{-12\pi a^5}{5}$$