BIT Calculus: Examination Review December 2014

## PART A. MULTIPLE CHOICE QUESTIONS.

1. The domain of the function $f(x) = \dfrac{1}{\sqrt{(x-1)(x-2)}}$ is

   a) $[1, 2]$, or equivalently, $\{x : 1 \leq x \leq 2\}$.

   b) $(1, 2)$, or equivalently, $\{x : 1 < x < 2\}$.

   c) $(-\infty, 1] \bigcup [2, +\infty)$, or equivalently, $\{x : x \leq 1 \text{ and } x \geq 2\}$.

   d) $(-\infty, 1) \bigcup (2, +\infty)$, or equivalently, $\{x : x < 1 \text{ and } x > 2\}$.

   e) None of the above.

   Answer: d)

2. Let $f(x) = 2x^3 - 1$ and $g(x) = \dfrac{1}{x}$. Then the value of the composition $f(g(1))$ is

   a) 1. b) $-\dfrac{1}{3}$. c) $-1$. d) $-3$. e) None of the above.

   Answer: a)

3. If $e^{x+2} = 3$, what is $x$ ?

   a) $\ln 2$. b) $3 - \ln 2$. c) $\ln 3 - 2$. d) $\dfrac{\ln 3}{2}$. e) None of the above.

   Answer : c)

4. The expression $\dfrac{(x^{0.4})^2 \cdot x^{-3.8}}{x^2}$ simplifies to

   a) $x^5$. b) $x^{-5}$. c) $x$. d) $x^{-1}$. (e) None of the above.

   Answer: (b)

5. Which of the following is equal to $\log_{\frac{1}{2}} \dfrac{1}{8}$ ?

   a) $\dfrac{1}{3}$. b) $-\dfrac{1}{3}$. c) 3. d) $-3$. e) None of the above.

   Answer: c)

6. Which of the following is equal to $e^{-2\ln 3}$ ?

a) $\frac{1}{9}$. b) $-\frac{1}{9}$. c) $-9$. d) 9. e) None of the above.

Answer: a)

7. The statement $3\ln(2x) - \ln(x^2) + \ln 5$ written as a single logarithm is

a) $\ln x$. b) $\ln(40x)$. c) $\ln(8x^3 - x^2 + 5)$. d) $\ln(5 - x^2)$. e) None of the above.

Answer: b)

8. What is $\lim_{x\to\infty} \frac{4x^3 + x^2 - 3}{1 + 2x - x^3}$ ?

a) $-4$. b) $-3$. c) 0. d) $\infty$. e) None of the above.

Answer: a)

9. What is the slope of the curve $y = 3x^{2/3}$ at $x = \frac{1}{8}$?

a) 1. b) 4. c) $\frac{1}{4}$. d) $\frac{4}{3}$. e) None of the above.

Answer: b)

10. Given that the profit function for a company is

$$P(x) = x^2 + \frac{50}{x} + 10x - 7,$$

find the **marginal profit** at a production level of $x = 5$ units.

a) 78. b) 87. c) 22. d) 18. e) None of the above.

Answer: d)

11. The derivative $f'$ of $f(x) = (3x^2 - e^x)^7$ is

a) $7(3x^2 - e^x)^6 \cdot (6x - e^x)$. b) $7(3x^2 - e^x)^6 \cdot (6x - 1)$. c) $(3x^2 - e^x)^6 \cdot (6x - e^x)$.

d) $7(3x^2 - e^x) \cdot (6x - 1)$. e) None of the above.

Answer: a)

12. What are the critical numbers of the function $f(x) = \dfrac{1}{x+2}$?

a) $-2$.  b) 0.  c) 2.  d) No critical numbers.  e) None of the above.

Answer: d)

13. The second derivative $f''$ of $f(x) = \ln(3x-1)$, $(x > \frac{1}{3})$ is

a) $\dfrac{-9}{(3x-1)^2}$.  b) $\dfrac{3}{(3x-1)^2}$.  c) $\dfrac{-9x}{3x-1}$.  d) $\dfrac{3}{3x-1}$.  e) None of the above.

Answer: a)

14. The graph of the function $y = \dfrac{x^2}{x^2+1}$ has

a) no horizontal asymptote and a vertical asymptote $x = -1$.

b) a horizontal asymptote $y = 1$ and a vertical asymptote $x = -1$.

c) a horizontal asymptote $y = 1$ and no vertical asymptote.

d) neither horizontal no vertical asymptote.

e) none of the above.

Answer: c)

15. Consider the function $f(x) = x^3 - 3x^2 + 1$. The function is increasing when:

a) $x \in (0, 2)$.

b) $x \in (0, \infty)$.

c) $x \in (-\infty, 0) \bigcup (2, \infty)$.

d) $x \in \mathbb{R}$.

e) None of the above.

Answer: c)

16. Let $f(x, y) = \frac{x^4}{6} - x^2 + \frac{5}{6}$. What are the inflection points of the function (if any)?

a) $(0, -1)$ and $(0, 1)$. b) $(-1, 0)$ and $(1, 0)$. c) $(1, 0)$.

d) No inflection points. e) None of the above.

Answer: b)

17. The graph of the function $f(x) = x^3 - 9x^2 + 12x + 23$ is concave downwards (DOWN) over the interval

a) $(3, +\infty)$. b) $(-3, 3)$. c) $(-\infty, 3)$. d) $(-\infty, +\infty)$.

Answer: c)

18. Suppose that a population of 300 species is growing **exponentially** with growth constant 0.5, where $t$ is measured in months. What is the size of the population in 4 months? (Choose the best approximation.)

a) 2217. b) 2483. c) 2011. d) 1946.

Answer: a)

19. The value of $\int_{-1}^{2} (x + 1)\, dx$ is

a) $-2.5$. b) $-1$. c) $3$. d) $4.5$. e) None of the above.

Answer: d)

20. The area of the region bounded by the $x$ axis and the parabola $y = -x^2 + 4$, between the lines $x = 0$ and $x = 3$, is represented by

a) $\int_0^2 (-x^2 + 4)\, dx - \int_2^3 (-x^2 + 4)\, dx$. b) $\int_0^3 (-x^2 + 4)\, dx$.

c) $\int_0^2 (-x^2 + 4)\, dx + \int_2^3 (-x^2 + 4)\, dx$. d) $\int_0^3 (x^2 - 4)\, dx$.

Answer: a)

**PART B. Answer all questions and show all appropriate steps in your work; otherwise only partial marks may be awarded.**

**B1.** An amount of \$10,000 is deposited in a bank that pays interest at the rate of 6% per year. Answer the questions below, rounding the answers to two decimal points.

**(a)** If the money is invested for the period of 5 year compounded **quarterly**, what would be the **interest** earned?

**(b)** How long would it take for the investment to grow to \$12,000 if the interest is compounded **semiannually**?

**(c)** What would be the accumulated amount (or return) after 10 years from the investment if the interest is compounded **continuously**?

**(d)** What should the interest rate be with **continuous compounding** in order to double the original amount of money after 10 years?

**Solution:**

**(a)** $A(t) = P\Big(1+\dfrac{r}{m}\Big)^{mt} = 10,000\Big(1+\dfrac{0.06}{4}\Big)^{4\cdot 5} = 10,000(1.015)^{20} = 13,468.55;$

$I = A - P = 3,468.55.$

**(b)** $A(t) = P\Big(1+\dfrac{r}{m}\Big)^{mt} \;\Rightarrow\; 12,000 = 10,000\Big(1+\dfrac{0.06}{2}\Big)^{2\cdot t} \;\Rightarrow\; 1.2 = 1.03^{2t}$

$\Rightarrow\; \ln(1.2) = 2t\ln(1.03) \quad \Rightarrow t = \dfrac{\ln(1.2)}{2\ln(1.03)} \approx 3.08.$

**(c)** $A(t) = Pe^{rt} = 10,000\, e^{0.06\cdot 10} = 18,221.19.$

**(d)**

$$A(t) = 2P = Pe^{r\cdot 10};\ \text{Solve for } r:\ e^{10r} = 2,\ 10r = \ln 2,\ r = \frac{\ln 2}{10} = 0.0693 = 6.93\%.$$

**B2.** Find the dimensions of the rectangular garden of the area 3600 $\mathrm{m}^2$ that can be fenced off (all four sides) with the minimum amount of fencing. (Assume that the sides $1 < x < 80, 1 < y < 80$.)

**Solution:**

Let $x$ and $y$ the the dimensions of the rectangle. Then its area $A = xy = 3600$ and the perimeter is $P = 2x + 2y$. Since $xy = 3600$, then $y = 3600/x$. Substituting $y = 3600/x$ into $P = 2x + 2y$ yields

$$P(x) = 2x + 2\cdot\frac{3600}{x} \;\to\; \text{min.}$$

To minimize a function of one variable, one must calculate its first derivative and find the critical numbers:

$$P'(x) = 2 - \frac{2\cdot 3600}{x^2} = \frac{2x^2 - 2\cdot 3600}{x^2}.$$

$P'(x)$ is not defined for $x = 0$ but $x = 0$ is not in the domain of $P$. $P'(x) = 0$ when $x = 60$. The function $P(x)$ has a local minimum at $x = 60$, which is also the absolute minimum of $f$ on $[1, 80]$. Indeed, $P(1) = 2 + 2\cdot 3600 = 7,202$, $P(80) = 160 + 2\cdot 45 = 250$, $P(60) = 120 + 2\cdot 60 = 240$. Thus, the desired dimensions are $x = 60$, $y = 3600/60 = 60$, so the garden must be a square.

**B3.**

**(a)** Evaluate the following integrals:

(i) $x > 0$, $\int \left(2x^5 + \frac{1}{x}\right) dx = 2\int x^5\,dx + \int \frac{1}{x}\,dx = 2 \cdot \frac{x^6}{6} + \ln x + C = \frac{x^6}{3} + \ln x + C.$

(ii) $\int_0^2 (3x-2)^3\,dx$

Let $u = 3x - 2$, then $\frac{du}{dx} = 3,\ du = 3\,dx,\ dx = \frac{du}{3}$.

Change the limits of integration: $x = 0 \Rightarrow u = 3x - 2 = -2, \quad x = 2 \Rightarrow u = 3x - 2 = 4$. Thus,

$$\int_0^2 (3x-2)^3\,dx = \int_{-2}^4 u^3 \frac{du}{3} = \frac{1}{3}\int_{-2}^4 u^3\,du = \frac{1}{3} \cdot \frac{u^4}{4}\Big|_{-2}^4 = \frac{1}{12}[4^4 - (-2)^4] = \frac{240}{12} = 20.$$

(iii) $\int 2e^{1-6x}\,dx = 2\int e^u \cdot \left(\frac{du}{-6}\right) = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{1-6x} + C$

(iv) $\int \frac{x+5}{x}\,dx = \int \left(\frac{x}{x} + \frac{5}{x}\right) dx = \int 1\,dx + 5\int \frac{1}{x}\,dx = x + 5\ln|x| + C.$

**(b)** Consider the equation $x^3 - 2xy + y^4 = 5$, where $y = y(x)$ is defined implicitly as a function of $x$. Find $\frac{dy}{dx}$.

Rewrite expression, stating that $y = y(x)$ is a function of $x$:

$x^3 - 2x \cdot y(x) + (y(x))^4 = 5.$

Differentiate both parts of the above expression with respect to $x$:

$3x^2 - 2 \cdot y(x) - 2x \cdot y'(x) + 4(y(x))^3 \cdot y'(x) = 0.$

Solve for $y'(x)$:

$-2x \cdot y'(x) + 4y^3 \cdot y'(x) = -3x^2 + 2y, \qquad y'(x)(4y^3 - 2x) = 2y - 3x^2, \quad y'(x) = \frac{2y - 3x^2}{4y^3 - 2x}.$