1-a $F(s) = \dfrac{s^3+3s^2+s+1}{s(s+1)(s+2)(s+3)} = \dfrac{a_1}{s} + \dfrac{a_2}{s+1} + \dfrac{a_3}{s+2} + \dfrac{a_4}{s+3}$ 1

$\boxed{a_1 = sF(s)\Big|_{s=0} = 1}$, $a_2 = (s+1)F(s)\Big|_{s=-1} = \dfrac{-1+3-1+1}{-1(1)(2)} = \boxed{-1}$

$a_3 = (s+2)F(s)\Big|_{s=-2} = \dfrac{-8+\overset{10}{12}-2+1}{(-2)(-1)(1)} = \boxed{\dfrac{3}{2}}$

$a_4 = (s+3)F(s)\Big|_{s=-3} = \dfrac{-27+27\overset{-2}{-3+1}}{-3(-2)(-1)} = \boxed{\dfrac{1}{3}}$

$F(s) = \dfrac{1}{s} - \dfrac{1}{s+1} + \dfrac{3}{2}\dfrac{1}{s+2} + \dfrac{1}{3}\dfrac{1}{s+3}$

$\boxed{f(t) = u(t) - e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) + \frac{1}{3}e^{-3t}u(t)}$

b- $F(s) = \dfrac{s^3+3s^2+s+1}{(s+1)(s+2)(s+3)} = \dfrac{s^3+3s^2+s+1}{s^3+(1+2+3)s^2+(2+3+6)s+6}$

$= \dfrac{s^3+3s^2+s+1}{s^3+6s^2+11s+6}$

$$\begin{array}{r|l} & 1 \\ \hline s^3+6s^2+11s+6 & s^3+3s^2+s+1 \\ & s^3+6s^2+11s+6 \\ \hline & -3s^2-10s-5 \end{array}$$

$F(s) = 1 - \dfrac{3s^2+10s+5}{(s+1)(s+2)(s+3)}$

$= 1 - \left(\dfrac{a_1}{s+1} + \dfrac{a_2}{s+2} + \dfrac{a_3}{s+3}\right)$

$a_1 = \dfrac{3(-1)^2+10(-1)+5}{(-1+2)(-1+2)} = \dfrac{3-10+5}{(1)(1)} = \boxed{-2}$

2

$$a_2 = \frac{3(-2)^2+10(-2)+5}{(-1)(1)} = \frac{12-20+5}{-1} = \boxed{3}$$

$$a_3 = \frac{3(-3)^2+10(-3)+5}{(-2)(-1)} = \frac{27-30+5}{2} = \boxed{16}$$

$$F(s) = 1 + \frac{2}{s+1} - \frac{3}{s+2} - \frac{16}{s+3}$$

$$\boxed{f(t) = \delta(t) + 2e^{-t}u(t) - 3e^{-2t}u(t) - 16e^{-3t}u(t)}$$

---

c- $$F(s) = \frac{s^3+3s^2+s+1}{(s+1)^2(s+3)} = \frac{s^3+3s^2+s+1}{s^3+(2+3)s^2+(6+1)s+3}$$

$$= \frac{s^3+3s^2+s+1}{s^3+5s^2+7s+3}$$

$$\begin{array}{r|l} & 1 \\ \hline s^3+5s^2+7s+3 & s^3+3s^2+s+1 \\ & s^3+5s^2+7s+3 \\ \hline & -2s^2-6s-2 \end{array}$$

$$F(s) = 1 - \underbrace{\frac{2s^2+6s+2}{(s+1)^2(s+3)}}_{G(s)} = 1 - \left(\frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \frac{a_3}{s+3}\right)$$

$$a_3 = \frac{2(-3)^2+6(-3)+2}{(-3+1)^2} = \frac{18-18+2}{4} = \boxed{\frac{1}{2}}$$

$$a_2 = \cancel{(s+1)^2 F(s)} \quad (s+1)^2 G(s)\Big|_{s=-1} = \frac{2s^2+6s+2}{s+3}\Big|_{s=-1} = \frac{2-6+2}{2} = \boxed{-1}$$

3

$$a_1 = \frac{d}{ds}(s+1)^2 G(s)\Big|_{s=-1} = \frac{d}{ds}\left(\frac{2s^2+6s+2}{s+3}\right)\Big|_{s=-1}$$

$$= \frac{(s+3)(4s+6)-(2s^2+6s+2)}{(s+3)^2}\Big|_{s=-1}$$

$$= \frac{4s^2+18s+18-2s^2-6s-2}{(s+3)^2}\Big|_{s=-1}$$

$$= \frac{2s^2+12s+16}{(s+3)^2}\Big|_{s=-1} = \frac{2-12+16}{2^2} = \boxed{\frac{3}{2}}$$

$$F(s) = 1 - \frac{3}{2}\frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{1}{2}\frac{1}{s+3}$$

$$\boxed{f(t) = \delta(t) - \frac{3}{2}e^{-t}u(t) + te^{-t}u(t) - \frac{1}{2}e^{-3t}ut}$$

---

d- $F(s) = \dfrac{3s^2+s+1}{s^2+5s+4}$

$$= 3 - \frac{14s+11}{s^2+5s+4}$$

$$= 3 - \frac{14s+11}{(s+1)(s+4)}$$

$$= 3 - \left(\frac{a_1}{s+1} + \frac{a_2}{s+4}\right)$$

$$\begin{array}{r|l} & 3 \\ s^2+5s+4 & 3s^2+s+1 \\ & 3s^2+15s+12 \\ \hline & -14s-11 \end{array}$$

$s^2+5s+4 = (s+4)(s+1)$

$$a_1 = \frac{14s+11}{s+4}\Bigg|_{s=-1} = \frac{-14+11}{3} = \boxed{-1}$$ 4

$$a_2 = \frac{14(-4)+11}{-4+1} = \boxed{15}$$

$$F(s) = 3 + \frac{1}{s+1} - \frac{15}{s+4}$$

$$\boxed{f(t) = 3\,\delta(t) + e^{-t}u(t) - 15e^{-4t}ut}$$

---

e- $F(s) = \dfrac{s^2+2s+12}{s(s^2+4s+5)}$ *

$$= \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+4s+5}$$

$$a_1 = sF(s)\Big|_{s=0} = \boxed{\frac{12}{5}}$$

$$F(s) = \frac{12}{5}\frac{1}{s} + \frac{a_2 s + a_3}{s^2+4s+5}$$

$$= \frac{\frac{12}{5}(s^2+4s+5) + (a_2 s + a_3)s}{s(s^2+4s+5)}$$

$$\cancel{= \left(\frac{12}{5}+a_2\right)s^2 + \frac{12(4)}{5}\ (a_2)(5)\left(\frac{12}{5}\right)}$$

$$= \frac{\left(\frac{12}{5}+a_2\right)s^2 + \left(\frac{12}{5}(4)+a_3\right)s + 12}{s(s^2+4s+5)} = \frac{\left(\frac{12}{5}+a_2\right)s^2 + \left(\frac{48}{5}+a_3\right)s + 12}{s(s^2+4s+5)}$$ **

Roots of:

$$s^2+4s+5 : \frac{-4 \pm \sqrt{(4)^2-4(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= -2 \pm j$$

$$s^2+4s+5$$

$$= (s+2)^2+1$$

$$= (s+\alpha)^2+\omega_0^2$$

$$\Rightarrow \alpha = 2,\ \omega_0 = 1$$

From (*) and (**)

$$s^2+2s+\cancel{12} \equiv \left(\frac{12}{5}+a_2\right)s^2+\left(\frac{48}{5}+a_3\right)s\cancel{+12}$$

$$\frac{12}{5}+a_2=1 \Longrightarrow a_2=\boxed{-\frac{7}{5}}$$

$$\frac{48}{5}+a_3=2 \Longrightarrow a_3=\boxed{-\frac{38}{5}}$$

$$F(s)=\frac{12}{5}\frac{1}{s}-\frac{\frac{7}{5}s+\frac{38}{5}}{s^2+4s+5}$$

$$=\frac{12}{5}\frac{1}{s}-\frac{1}{5}\frac{7(s+2)+24}{(s+2)^2+1}$$

$$=\frac{12}{5}\frac{1}{s}-\frac{7}{5}\frac{s+2}{(s+2)^2+1}-\frac{24}{5}\frac{1}{(s+2)^2+1}$$

$$\boxed{f(t)=\frac{12}{5}u(t)-\left(\frac{7}{5}e^{-2t}\cos t+\frac{24}{5}e^{-2t}\sin t\right)u(t)}$$

---

f- $$F(s)=\frac{1}{s(s+1)^2(s^2+4s+5)} \quad (***)$$

$$=\frac{a_1}{s}+\frac{a_2}{s+1}+\frac{a_3}{(s+1)^2}+\frac{a_4s+a_5}{s^2+4s+5}$$

From part (e), we know that s^2+4s+5 has complex roots and that $s^2+4s+5 = (s+2)^2+1$

$$a_1 = sF(s)\Big|_{s=0} = \boxed{\frac{1}{5}}$$

$$a_3 = (s+1)^2 F(s)\Big|_{s=-1} = \frac{1}{(-1)(1-4+5)} = \boxed{-\frac{1}{2}}$$

$$a_2 = \frac{d}{ds}(s^2+1)F(s)\Big|_{s=-1} = \frac{d}{ds}\left(\frac{1}{s(s^2+4s+5)}\right)\Big|_{-1}$$

$$= -\frac{s^2+4s+5 + s(2s+4)}{s^2(s^2+4s+5)^2}\Big|_{s=-1}$$

$$= -\frac{3s^2+8s+5}{s^2(s^2+4s+5)^2}\Big|_{s=-1}$$

$$= -\left(\frac{3-8+5}{(1)(1-4+5)^2}\right) = \boxed{0}$$

$$F(s) = \frac{1}{5s} - \frac{1/2}{(s+1)^2} + \frac{a_4 s + a_5}{s^2+4s+5} \qquad ****$$

From *** and ****

$$\frac{1}{5}(s+1)^2(s^2+4s+5) - \frac{1}{2}s(s^2+4s+5) + (a_4 s + a_5)s(s+1)^2 = 1$$

$$\left(\frac{1}{5} + a_4\right)s^4 + \left(\frac{1}{5}(2+4) - 2 + 2a_4 + a_5\right)s^3 + \left(\frac{1}{5}(5+8+1) - 8 + (a_4 + 2a_5)\right)s^2$$

$$+ \left(\frac{1}{5}(4+10) - 10 + a_5\right)s + \frac{1}{5}(5) = 1$$

Coefficient of $s^4 = 0$

$$\Rightarrow \left(\frac{1}{5} + a_4\right) = 0 \quad \text{or} \quad a_4 = -\frac{1}{5}$$

Coefficient of $s = 0$

$$a_5 = 10 - \frac{14}{5} = \frac{36}{5}$$

Check: coefficients of s^2 and $s^3 = 0$

$$\frac{1}{5}(s+1)^2(s^2+4s+5) - \frac{1}{2}s(s^2+4s+5) + (a_4 s + a_5)s(s+1)^2 \equiv 1$$

Coefficient of s^4: $\frac{1}{5} + a_4 = 0 \implies a_4 = -\frac{1}{5}$
$= 0$

Coefficient of s : $\frac{1}{5}(10+4) - \frac{5}{2} + a_5 = 0$

$$\implies a_5 = \frac{5}{2} - \frac{14}{5} = \frac{25-28}{10} = \frac{-3}{10}$$

For check, consider the coefficients of s^2 and s^3

Coefficient of s^2: $\frac{1}{5}(5+8+1) - \frac{1}{2}(4) + (a_4 + 2a_5)$

$$= \frac{1}{5}(14) - 2 + \left(-\frac{1}{5}\right) - \frac{2(3)}{10}$$

$$= \frac{4}{5} - \frac{1}{5} - \frac{3}{5} = \underline{\underline{0}} \quad \text{(Correct)}$$

$\boxed{2}$

$$F(s) = \frac{1}{s^2}(1 - e^{-2s}) - \frac{1}{s}e^{-2s}$$

$f(t)u(t) \xrightarrow{\mathcal{L}} F(s)$

$f(t-\alpha)u(t-\alpha) \longrightarrow F(s)e^{-\alpha s}$

$t\,u(t) \longrightarrow \frac{1}{s^2}$

$u(t) \longrightarrow \frac{1}{s}$

$$F(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-2s}$$

$$\implies \boxed{f(t) = t\,u(t) - (t-2)u(t-2) - u(t-2)}$$

3] a- $\ddot{x} + \dot{x} + x = \cos 2t \qquad x(0) = 0\ ,\ \dot{x}(0) = 0$ (8)

$$s^2 X(s) - s x(0) - \dot{x}(0) + s X(s) - x(0) + X(s) = \frac{s}{s^2+4}$$

$$\cancel{(s^2+s+1)} X(s) = \frac{s}{(s^2+4)(s^2+s+1)} \qquad (*)$$

s^2+s+1 has complex roots: $s^2+s+1 = (s+\frac{1}{2})^2 + \frac{3}{4}$

$(\alpha = \frac{1}{2},\ \omega = \frac{\sqrt{3}}{2})$

$$X(s) = \frac{a_1 s + b_1}{s^2+4} + \frac{a_2 s + b_2}{s^2+s+1}$$

$$= \frac{(a_1 s + b_1)(s^2+s+1) + (a_2 s + b_2)(s^2+4)}{(s^2+4)(s^2+s+1)} \qquad (**)$$

Comparing $(*)$ and $(**)$

$$s \equiv (a_1+a_2)s^3 + (a_1+b_1+a_2+b_2)s^2 + (a_1+b_1+4a_2)s + b_1 + 4b_2$$

① $\boxed{a_1 + a_2 = 0}$, $\boxed{a_1 + b_1 + a_2 + b_2 = 0}$ ② , $\boxed{a_1 + b_1 + 4a_2 = 1}$ ③

$\boxed{b_1 + 4b_2 = 0}$ ④

From ① and ② $a_2 = -a_1$ and $b_2 = -b_1$

substituting in ③ and ④ $\Longrightarrow a_1 - 4a_1 + b_1 = 1 \Longrightarrow b_1 - 3a_1 = 1$

$\Longrightarrow b_1 - 4b_1 = 0 \Longrightarrow b_1 = 0 \longrightarrow a_1 = -1/3$

$b_2 = 0 \qquad a_2 = 1/3$

$$X(s) = \frac{-\frac{1}{3}\, s}{s^2+4} + \frac{1}{3}\,\frac{s}{s^2+s+1}$$

$$= \frac{-\frac{1}{3}\, s}{s^2+4} + \frac{1}{3}\,\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{-\frac{1}{3}\, s}{s^2+4} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{6}\,\frac{2}{\sqrt{3}}\,\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$x(t) = \left(-\frac{1}{3}\cos 2t + \frac{1}{3}\cos\frac{\sqrt{3}}{2}t\, e^{-\frac{1}{2}t} - \frac{1}{3\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\, e^{-\frac{1}{2}t}\right)u(t)$$

---

b] $\ddot{x} + 4x = t$, $x(0)=0$, $\dot{x}(0)=0$

$$s^2X(s) - s\,x(0) - \dot{x}(0) + 4X(s) = \frac{1}{s^2}$$

$$X(s) = \frac{1}{s^2(s^2+4)} = \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3 s + a_4}{s^2+4}$$

$$a_2 = s^2X(s)\Big|_{s=0} = \boxed{\frac{1}{4}}$$

$$a_1 = \frac{d}{ds}s^2X(s)\Big|_{s=0} = \frac{d}{ds}(s^2+4)^{-1}\Big|_{s=0} = -(s^2+4)^{-2}\,2s\Big|_{s=0} \boxed{=0}$$

$$X(s) = \frac{1}{4s^2} + \frac{a_3 s + a_4}{s^2+4} = \frac{\frac{1}{4}(s^2+4) + (a_3 s + a_4)s^2}{s^2(s^2+4)}$$

Comparing coefficients of numerator:

$$a_3 s^3 + \left(\frac{1}{4} + a_4\right) s^2 + 1 \equiv 1 \Rightarrow \boxed{a_3 = 0, \quad a_4 = -\frac{1}{4}}$$

$$X(s) = \frac{1}{4s^2} - \frac{1}{4\,(s^2+4)} = \frac{1}{4s^2} - \frac{1}{4} \times \frac{1}{2}\,\frac{2}{s^2+4}$$

$$x(t) = \frac{1}{4} t\,u(t) - \frac{1}{4} \times \frac{1}{2} \sin 2t\;u(t)$$

---

c) $\ddot{x} + 4x = t, \qquad x(0) = 0, \quad \dot{x}(2) = 0$

$$s^2 X(s) - \cancel{s\,x(0)} - \dot{x}(0) + 4X(s) = \frac{1}{s^2}$$

$$(s^2+4)X(s) = \frac{1}{s^2} \cancel{+s} + \dot{x}(0)$$

$$X(s) = \frac{1 + \dot{x}(0)\,s^2}{s^2+4} = \dot{x}(0) + \frac{1 - 4\dot{x}(0)}{s^2+4}$$

$$\begin{array}{r|l} & \dot{x}(0) \\ s^2+4 & \dot{x}(0)s^2 + 1 \\ & \dot{x}(0)s^2 + \dot{x}(0)4 \\ \hline & 1 - 4\dot{x}(0) \end{array}$$

$$\Rightarrow x(t) = \dot{x}(0)\,\delta(t) + \frac{1-4\dot{x}(0)}{2} \sin 2t, \quad t \geq 0$$

$$\dot{x}(t) = \dot{x}(0)\frac{d\delta(t)}{dt} + \frac{1-4\dot{x}(0)}{\cancel{2}} \cos 2t \cdot \cancel{2}$$

$$\dot{x}(2) = 0 + (1 - 4\dot{x}(0)) \cos 4 = 0$$

$\Rightarrow$ ~~$\frac{1}{4} \dot{x}(0)$~~ ~~$\dot{x}(0) = \frac{1}{\cos 4}$~~

$$\dot{x}(0) = \frac{1}{4}\left(1 - \frac{1}{\cos 4}\right)$$

$$x(t) = \frac{1}{4}\left(1 - \frac{1}{\cos 4}\right) \delta(t) + \frac{1}{\cos 4} \sin 2t \,, \quad \text{for } t \geq 0$$

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