9.1 Consider the following second-order ODE:

$$\frac{d^2y}{dx^2} = y + x^2$$

(*a*) Using the central difference formula for approximating the second derivative, discretize the ODE (rewrite the equation in a form suitable for solution with the finite difference method).

(*b*) If the step size is $h = 1$, what is the value of the diagonal elements in the resulting matrix of coefficients of the system of linear equations that has to be solved?

Solution

(*a*) The central difference formula for the second derivative at the point $x = x_i$ is given by $\frac{d^2y}{dx^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$. Substituting this into the ODE yields:

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = y_i + x_i^2$$

Collecting like terms gives:

$$y_{i-1} - (2 + h^2)y_i + y_{i+1} = h^2x_i^2$$

which is the discretized form of the ODE.

(*b*) The discretized form of the ODE obtained in part (*a*) is a tridiagonal system of equations with the *y*s as the unknowns. The diagonal elements are the coefficients of the y_i terms: $-(2 + h^2)$. For $h = 1$, the diagonal elements are equal to -3.

9.2 Consider the following second-order ODE:

$$\frac{d^2y}{dr^2} + \frac{y}{r} = C$$

where C is a constant.

(a) Using the central difference formula for approximating the second derivative, discretize the ODE (rewrite the equation in a form suitable for solution with the finite difference method).

(b) If the step size is $h = 0.5$, what is the value of the diagonal elements in the resulting matrix of coefficients of the system of linear equations that has to be solved?

Solution

(a) The central difference formula for the second derivative at the point $x = x_i$ is given by $\frac{d^2y}{dx^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$. Substituting this into the ODE yields:

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{y_i}{r_i} = C$$

Collecting like terms gives:

$$r_i y_{i-1} + (h^2 - 2r_i)y_i + r_i y_{i+1} = Ch^2 r_i$$

which is the discretized form of the ODE.

(b) The discretized form of the ODE obtained in part (a) is a tridiagonal system of equations with the ys as the unknowns. The diagonal elements are the coefficients of the y_i terms: $h^2 - 2r_i$. For $h = 0.5$, the diagonal elements are equal to $0.25 - 2r_i$.

9.3 Consider the following second-order ODE:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 2x \text{ for } 0 \le x \le 1 \text{ , with } y(0) = 1 \text{ and } y(1) = 1$$

(a) Using the central difference formulas for approximating the derivatives, discretize the ODE (rewrite the equation in a form suitable for solution with the finite difference method).

(b) What is the expression for the diagonal terms in the resulting matrix of coefficients of the tridiagonal system of linear equations that has to be solved?

Solution

(a) The central difference formula for the second derivative at the point $x = x_i$ is given by $\frac{d^2y}{dx^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$. The central difference formula for the first derivative at the point $x = x_i$ is given by $\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h}$. Substituting these into the ODE yields:

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + x_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + y_i = 2x_i$$

Collecting like terms, the discretized form of the ODE is:

$$(2 - hx_i)y_{i-1} + (2h^2 - 4)y_i + (2 + hx_i)y_{i+1} = 4x_ih^2$$

This forms the following system of equations:

$$\begin{bmatrix} (2-hx_2) & (2h^2-4) & (2+hx_2) & 0 & \dots & 0 & 0 \\ 0 & (2-hx_3) & (2h^2-4) & (2+hx_3) & \dots & 0 & 0 \\ 0 & 0 & (2-hx_4) & (2h^2-4) & (2+hx_4) & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & (2-hx_{n-1}) & (2h^2-4) & (2+hx_{n-1}) & 0 \\ 0 & 0 & \dots & 0 & (2-hx_n) & (2h^2-4) & (2+hx_n) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \dots \\ y_{n-1} \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 4x_2h^2 \\ 4x_3h^2 \\ 4x_4h^2 \\ \dots \\ \dots \\ 4x_{n-1}h^2 \\ 4x_nh^2 \end{bmatrix}$$

Note that $y_1 = y(0) = 1$ and $y_{n+1} = y(1) = 1$. Substituting into the above system and rewriting the first and last equations (rows) yields the following tridiagonal system of equations:

$$\begin{bmatrix} (2h^2-4) & (2+hx_2) & 0 & \dots & 0 & 0 & 0 \\ (2-hx_3) & (2h^2-4) & (2+hx_3) & 0 & \dots & 0 & 0 \\ 0 & (2-hx_4) & (2h^2-4) & (2+hx_4) & 0 & \dots & 0 \\ \dots & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & (2-hx_{n-1}) & (2h^2-4) & (2+hx_{n-1}) \\ & 0 & 0 & \dots & 0 & (2-hx_n) & (2h^2-4) \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ \dots \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 4x_2h^2-2+hx_2 \\ 4x_3h^2 \\ 4x_4h^2 \\ \dots \\ 4x_{n-1}h^2 \\ 4x_nh^2-2-hx_n \end{bmatrix}$$

(*b*) From part (*a*) above, it can be seen from the resulting tridiagonal matrix that the above-diagonal terms are: $(2+hx_i)$, where $i = 2, 3, \dots n-1$.

9.4 Consider the following boundary value problem:

$$\frac{d^2y}{dx^2} + ay + by^4 = 0 \text{ for } 0 \le x \le 1 \text{, with the boundary conditions: } \left.\frac{dy}{dx}\right|_{x=0} = 0 \text{ and } y(1) = 1$$

where a and b are constants. Discretize the second-order ODE using:
(a) Second-order accurate forward difference.
(b) Second-order accurate backward difference.
(c) Discretize the boundary condition at $x = 0$ using the second-order accurate forward difference.

Solution

(a) The second-order accurate forward difference formula for the second derivative is the four-point formula given in Table 6-1, $f''(x_i) = \frac{2f(x_i) - 5f(x_{i+1}) + 4f(x_{i+2}) - f(x_{i+3})}{h^2}$, which in the present case would be written as: $\frac{d^2y}{dx^2} = \frac{2y_i - 5y_{i+1} + 4y_{i+2} - y_{i+3}}{h^2}$. Substituting into the ODE yields the following discretized form:

$$\frac{2y_i - 5y_{i+1} + 4y_{i+2} - y_{i+3}}{h^2} + ay_i + by_i^4 = 0$$

Note that $y_{n+1} = y(1) = 1$.

(b) The second order backward difference formula for the second derivative is the four-point formula $f''(x_i) = \frac{-f(x_{i-3}) + 4f(x_{i-2}) - 5f(x_{i-1}) + 2f(x_i)}{h^2}$, which in the present case would be written as $\frac{d^2y}{dx^2} = \frac{-y_{i-3} + 4y_{i-2} - 5y_{i-1} + 2y_i}{h^2}$. Substituting into the ODE yields the following discretized form:

$$\frac{-y_{i-3} + 4y_{i-2} - 5y_{i-1} + 2y_i}{h^2} + ay_i + by_i^4 = 0$$

(c) The boundary condition at $x = 0$ discretized using a second-order accurate difference formula, i.e. the three-point forward difference formula, yields $\left.\frac{dy}{dx}\right|_{x=0} = \frac{-3y_1 + 4y_2 - y_3}{2h} = 0$. Since $y_1 = y(0)$, is unknown, the latter equation can be solved to yield:

$$y_1 = \frac{y_3}{3} - \frac{4y_2}{3}$$

9.5 Consider the following boundary value problem:

$\frac{d^3y}{dx^3} + \frac{dy}{dx} - \sin y = 0$ for $0 \le x \le 1$, with the boundary conditions: $y(0) = 0$, $\left.\frac{dy}{dx}\right|_{x=0} = 0$ and $y(1) = 10$.

Discretize the third-order ODE using second-order central differences. When the boundary conditions are discretized, make sure that the order of the truncation error is compatible with that of the ODE.

Solution

The second-order accurate central difference formula for the third derivative is given in Table 6-1 as $f'''(x_i) = \frac{-f(x_{i-2}) + 2f(x_{i-1}) - 2f(x_{i+1}) + f(x_{i+2})}{2h^3}$, which in the present case may be written as $\frac{d^3y}{dx^3} = \frac{-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}}{2h^3}$. Similarly, the second order accurate central difference formula for the first derivative is $\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h}$. Substituting these into the ODE yields the following discretized equation:

$$\frac{-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}}{2h^3} + \frac{y_{i+1} - y_{i-1}}{2h} - \sin y_i = 0$$

Taking $y_1 = y(0) = 0$ and $y_{n+1} = y(1) = 10$, the only boundary condition that must be discretized is $\left.\frac{dy}{dx}\right|_{x=0} = 0$. If the second order accurate central difference formula is used, $\frac{y_2 - y_0}{2h} = 0$ or $y_0 = y_2$, where y_0 is an additional point that is introduced ***outside the original domain***. Note that y_0 will appear in the equation relating $y_0, y_1, y_2, y_3,$ and y_4. But, this is ***not*** useful because it introduces another unknown without providing another equation. Consequently, the derivative boundary condition must be discretized using the second order accurate forward difference formula $\left.\frac{dy}{dx}\right|_{x=0} = \frac{-3y_1 + 4y_2 - y_3}{2h} = \frac{4y_2 - y_3}{2h} = 0$ or $4y_2 - y_3 = 0$. Incorporating these boundary conditions, the system of equations that results from the discretized form of the ODE is:

$$4y_2 - y_3 = 0$$

$$\frac{-y_1 + 2y_2 - 2y_4 + y_5}{2h^3} + \frac{y_4 - y_2}{2h} - \sin y_3 =$$

$$\frac{-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}}{2h^3} + \frac{y_{i+1} - y_{i-1}}{2h} - \sin y_i = 0 \quad \text{for } i = 4, 5, \ldots, n-2$$

$$\frac{-y_{n-3}+2y_{n-2}-2y_n+y_{n+1}}{2h^3}+\frac{y_n-y_{n-2}}{2h}-\sin y_{n-1} = 0$$

Substituting for y_1 and y_{n+1} yields the following simplified system of equations:

$$4y_2-y_3 = 0$$

$$\frac{2y_2-2y_4+y_5}{2h^3}+\frac{y_4-y_2}{2h}-\sin y_3 = 0$$

$$\frac{-y_{i-2}+2y_{i-1}-2y_{i+1}+y_{i+2}}{2h^3}+\frac{y_{i+1}-y_{i-1}}{2h}-\sin y_i = 0 \text{ for } i = 4, 5, \ldots, n-2$$

$$\frac{-y_{n-3}+2y_{n-2}-2y_n+10}{2h^3}+\frac{y_n-y_{n-2}}{2h}-\sin y_{n-1} = 0$$

This is a system of $n-1$ equations in the $n-1$ unknowns y_2 through y_{n} .

9.6 Consider the following boundary value problem.

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = -Q$ for $0 \le x \le 1$, with the boundary conditions: $\left.\frac{dy}{dx}\right|_{x=0} = 0$ and $\left.\frac{dy}{dx}\right|_{x=1} = ay(1) + by^4(1)$.

where a and b are constants. Discretize the ODE using second-order accurate central differences for the derivatives. When the boundary conditions are discretized, make sure that the order of the truncation error is compatible with that of the ODE.

Solution

Using second order accurate central differences,

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{1}{x_i}\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) = -Q$$

Collecting like terms,

$$(2x_i - h)y_{i-1} - 4x_i y_i + (2x_i + h)y_{i+1} = -2Qh^2 x_i$$

Note that the entire equation has been multiplied by x_i to accommodate the singularity at $x = 0$. The boundary conditions are also discretized using the one-sided, second-order accurate difference formulae given in Table 6-1:

$$\frac{-3y_1 + 4y_2 - y_3}{2h} = 0$$

where y_1 is the value of y at $x = 0$, y_2 is the value of y at $x = h$, and y_3 is the value of y at $x = 2h$. This boundary condition can be simplified to:

$$y_1 = \frac{4y_2 - y_3}{3}$$

The second order backward difference formula is used for the other boundary condition:

$$\frac{y_{N-2} - 4y_{N-1} + 3y_N}{2h} = ay_N + by_N^4$$

Collecting terms,

$$y_{N-2} - 4y_{N-1} + (3 - 2ah)y_N - 2hby_N^4 = 0$$

Note that since y_N is unknown, and the associated boundary condition at $x = x_N$ is nonlinear, the solution methods described in Chapter 3 will have to be used in solving this set of equations.

9.7 Consider the following boundary value problem:

$$\frac{d^2u}{dx^2} - \pi^2 u = 2\pi^2 \sin(\pi x) \text{ for } 0 \le x \le 1 \text{, with the boundary conditions: } u(0) = 1 \text{ and } u(1) = -1$$

What are the diagonal elements of the resulting tridiagonal matrix when the finite difference method with first-order accurate central differences is applied to solve the problem with a step size of 1/4?

Solution

The discretized form of the ODE is:

$$-\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right) + \pi^2 u_i = 2\pi^2 \sin(\pi x_i)$$

Multiplying through by h^2 and collecting like terms yields:

$$-u_{i-1} + (2 + h^2\pi^2)u_i - u_{i+1} = 2\pi^2 h^2 \sin(\pi x_i)$$

The diagonal elements of this tridiagonal system are $(2 + h^2\pi^2)$. For a step size of $h = \frac{1}{8}$, the diagonal elements are $\left(2 + \frac{\pi^2}{64}\right) = 2.154213$.

9.8 Consider the second-order ODE of the form:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$$

where p and q are constants, and $r(x)$ is a given function. Using second-order accurate central differences for the derivatives, discretize the ODE.

Solution

Using Table 6-1,

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + p\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + qy_i = r(x_i)$$

Multiplying through by h^2 and collecting like terms,

$$(2 - ph)y_{i-1} + (2 + ph)y_{i+1} + (2qh^2 - 4)y_i = 2h^2r(x_i)$$

9.9 Given the boundary value problem $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ with $\left.\frac{dy}{dx}\right|_{x=0} = 0$ and $y(1) = 1$. Set up the problem for solving using the shooting method. Use the bisection method to determine the value of $y(0)$ such that the boundary condition at $x = 1$ is met within a specified error defined as $E_H = |y_c(1) - 1|$ where $y_c(1)$ is the calculated value of the solution at $x = 1$. Set up the equations to be solved but do not solve.

Solution

First, re-write the ODE as two first order ODEs:

$$\frac{dy}{dx} = w \qquad \text{with} \quad y(0) = Y_1$$

$$\frac{dw}{dx} = 2w - y \qquad \text{with} \quad w(0) = 0$$

where Y_1 is the initial guess for y at x=0. Solve the system to find $y_c(1)$. Suppose the value Y_1 is such that the calculated value $y_c(1)$ is below 1. Next, find a value Y_2 such that integrating the two ODEs with $y(0)=Y_2$ yields a calculated value of $y_c(1)$ above 1. Then set:

$$Y_3 = \frac{Y_1 + Y_2}{2}$$

and apply the bisection method until $|y_c(1)-1|<E_H$.

9.10 Given the boundary value problem $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ with $y(0) = 0$ and $\left.\frac{dy}{dx}\right|_{x=1} = 5$. Discretize the problem using second-order accurate central differences for the derivatives and a second-order accurate one-sided difference for the boundary condition at $x = 1$. For a constant step size h, what is the term on the right hand side of the last of the simultaneous equations that result?

Solution

Using Table 6-1, the central difference formulae for the derivatives yields:

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - 2\left[\frac{y_{1+1} - y_{i-1}}{2h}\right] + y_i = 0$$

Collecting like terms,

$$\left(\frac{1}{h^2} + \frac{1}{h}\right)y_{i-1} + \left(1 - \frac{2}{h^2}\right)y_i + \left(\frac{1}{h^2} - \frac{1}{h}\right)y_{i+1} = 0$$

The discretized boundary conditions are $y_i(0)=0$ and $\frac{y_{i-2} - 4y_{i-1} + 3y_i}{2h} = 5$. If the points are labeled $x_1=0$, x_2, x_3,…, $x_n=1$. Thus, $y_1=0$ and $\frac{y_{n-2} - 4y_{n-1} + 3y_n}{2h} = 5$. Solving for y_n:

$$y_n = \frac{10h - y_{n-2} + 4y_{n-1}}{3}$$

i=2:

$$\left(1 - \frac{2}{h^2}\right)y_2 + \left(\frac{1}{h^2} - \frac{1}{h}\right)y_3 = 0$$

i=3,4,…, n-2:

$$\left(\frac{1}{h^2} + \frac{1}{h}\right)y_{i-1} + \left(1 - \frac{2}{h^2}\right)y_i + \left(\frac{1}{h^2} - \frac{1}{h}\right)y_{i+1} = 0$$

i=n-1:

$$\left(\frac{2}{3h^2} + \frac{4}{3h}\right)y_{n-2} + \left(1 - \frac{2}{3h^2} - \frac{4}{3h}\right)y_{n-1} + \left(\frac{1}{h} - 1\right)\frac{(10)}{3} = 0$$

Therefore, the resulting linear system is:

$$\begin{bmatrix} \left(1-\frac{2}{h^2}\right) & \left(\frac{1}{h^2}-\frac{1}{h}\right) & 0 & \cdots & \cdots\ \cdots & 0 & 0 \\ \left(\frac{1}{h^2}+\frac{1}{h}\right) & \left(1-\frac{2}{h^2}\right) & \left(\frac{1}{h^2}-\frac{1}{h}\right) & 0 & \cdots\ \cdots & 0 & 0 \\ 0 & \left(\frac{1}{h^2}+\frac{1}{h}\right) & \left(1-\frac{2}{h^2}\right) & \left(\frac{1}{h^2}-\frac{1}{h}\right) & 0\ \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots\ \ddots & & \\ & & & & & & \\ 0 & \cdots & & & & \left(\frac{2}{3h^2}+\frac{4}{3h}\right) & \left(1-\frac{2}{3h^2}-\frac{4}{3h}\right) \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \\ \\ \frac{10}{3}\left(\frac{1}{h}-1\right) \end{bmatrix}$$

The term on the right hand side of the last equation is $\frac{10}{3}\left(\frac{1}{h}-1\right)$

9.11 Write the BVP in Problem 9.10 as a set of two first-order ODEs, ready to be solved using the shooting method. Use the linear interpolation method described in Section 9.2 to set up the equation for determining dy/dx at $x = 0$ such that the boundary condition at $x = 1$ may be satisfied to a desired error defined as $E_H = \left| \left.\frac{dy_c}{dx}\right|_{x=1} - 5 \right|$, where $\left.\frac{dy_c}{dx}\right|_{x=1}$ is the calculated value of the derivative at $x = 1$. Set up the equations to be solved but do not solve.

Solution

First, re-write the ODE as two first order ODEs:

$$\frac{dy}{dx} = w \qquad \text{with} \quad y(0) = 0$$

$$\frac{dw}{dx} = 2w - y \qquad \text{with} \quad w(0) = W_1$$

where W_1 is the initial guess for the slope of y at x=0. Solve the system to find $w_1(1)$ and $y_1(1)$. Suppose the value W_1 is such that the calculated value $w_1(1)$ is below 5. Next, find a value W_2 such that integrating the two ODEs with $w(0)=W_2$ yields $y_2(1)$ and a calculated value of $w_2(1)$ above 5. Then set:

$$W_n = W_1 + (y_c(1) - y_1(1))\frac{(W_2 - W_1)}{(y_2(1) - y_1(1))}$$

The new value W_n is then used in the next calculation with a new solution at the endpoint until $|w_n(1) - 5| \le E_H$.