MAT 1341F Test 4

Fall 2015

November 30, 4pm

Professor: Charles Starling

Family Name: Key.

First Name: ______

Student Number: ______

Enter Multiple Choice Answers Here	
1	C
2	C
3	D

DGD TA: ☐ Evelyn ☐ Cameron

Marker's Use Only	
4	
5	
6	
7 [Bonus]	
Total	

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY.

1. You have 80 minutes to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All cybernetic implants not necessary for life-support must be disabled at the beginning of the exam.
3. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 4 to 7, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
4. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. **Please record your answers in the table above.**
5. Questions 4 to 6 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
6. Question 7 is a challenging bonus question and is worth 3 points. It is *much* more difficult to obtain marks in the bonus question, so spend your time accordingly. You can earn 100% without attempting Q.7.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. Let A be a 5×4 matrix. Recall that $\ker(A) = \{x \in \mathbb{R}^4 \mid Ax = 0\}$ and $\operatorname{col}(A) = \{Ax \mid x \in \mathbb{R}^4\}$. If $\operatorname{rank}(A) = 3$, then

   A. $\operatorname{col}(A) = \mathbb{R}^5$ and $\dim(\ker(A)) = 1$

   B. $\operatorname{col}(A) = \mathbb{R}^5$ and $\dim(\ker(A)) = 4$

   C. $\dim(\operatorname{col}(A)) = 3$ and $\dim(\ker(A)) = 1$

   D. $\dim(\operatorname{col}(A)) = 3$ and $\dim(\ker(A)) = 2$

   E. $\dim(\operatorname{col}(A)) = 4$ and $\ker(A) = \mathbb{R}^4$

   F. $\operatorname{col}(A) = \{0\}$ and $\dim(\ker(A)) = 4$

m n

= rank(

$n = \dim(\ker(A)) + \dim(\text{col}(A))$

$4 = \dim(\ker(A)) + 3$

$\therefore \dim(\ker(A)) = 1$

2. Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. The third row of B^{-1} is:

   A. $[1 \;\; 2 \;\; 1]$

   B. $[-1 \;\; 1 \;\; 1]$

   C. $[-3 \;\; 1 \;\; 2]$

   D. $[0 \;\; 0 \;\; 1]$

   E. $[1 \;\; 1 \;\; -1]$

   F. B is not invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right]$$

3. Let $S = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and consider the subset $W = \{A \in \mathbf{M}_{22}(\mathbb{R}) \mid SA = AS\}$. Which one of the following statements is true?

A. W is not a subspace of $\mathbf{M}_{22}(\mathbb{R})$

B. W is a subspace of $\mathbf{M}_{22}(\mathbb{R})$, and $\dim(W) = 4$

C. W is a subspace of $\mathbf{M}_{22}(\mathbb{R})$, and $\dim(W) = 3$

(D.) W is a subspace of $\mathbf{M}_{22}(\mathbb{R})$, and $\dim(W) = 2$

E. W is a subspace of $\mathbf{M}_{22}(\mathbb{R})$, and $\dim(W) = 1$

F. W is a subspace of $\mathbf{M}_{22}(\mathbb{R})$, and $\dim(W) = 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$SA: \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+c & b+d \end{bmatrix}$$

$$AS: \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} b & b \\ d & d \end{bmatrix}$$

$AS = SA \Rightarrow b = 0, \quad a + c = d$

$$W = \left\{ \begin{bmatrix} a & 0 \\ c & a+c \end{bmatrix} \middle| \, a, c \in \mathbb{R} \right\}.$$

$$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

These are L.I., so $\dim(W) = 2$.

$w_1 \quad w_2 \quad w_3$

4. Let $U = \text{span}\{(1,0,1,0),(1,0,0,0),(0,1,0,-1)\}$.

   (a) Use the Gram Schmidt algorithm to find an orthogonal basis of U.

   (b) Find the best approximation to $(1,\ -1,\ 2,\ -1)$ by vectors in U.

   (c) Extend your basis in (a) to a basis of $\mathbb{R}^4$.

a) $v_1 = w_1 = (1,0,1,0)$

$v_2 = w_2 - \text{proj}_{v_1}(w_2)$

$= (1,0,0,0) - \dfrac{(1,0,1,0)\cdot(1,0,0,0)}{1^2+0^2+\not{0}^2+0^2}(1,0,1,0)$

$= (1,0,0,0) - \frac{1}{2}(1,0,1,0)$

$= \left(\frac{1}{2}, 0, -\frac{1}{2}, 0\right)$.

$v_3 = w_3 - \text{proj}_{v_1}(w_3) - \text{proj}_{v_2}(w_3)$

$= (0,1,0,-1) - \dfrac{v_1\cdot w_3}{\|v_1\|^2}v_1 - \dfrac{v_2\cdot w_3}{\|v_2\|^2}v_2$ $\quad v_1 \perp w_3,\ v_2 \perp w_3$ so both last terms are zero

$= (0,1,0,-1) \Rightarrow \{\overset{v_1}{(1,0,1,0)}, \overset{v_2}{(\tfrac{1}{2},0,-\tfrac{1}{2},0)}, \overset{v_3}{(0,1,0,-1)}\}$ is an orth. basis

b) $\text{proj}_U \underset{v}{(1,-1,2,-1)} = \dfrac{v\cdot v_1}{\|v_1\|^2}v_1 + \dfrac{v\cdot v_2}{\|v_2\|^2}v_2 + \dfrac{v\cdot v_3}{\|v_3\|^2}v_3$

$= \dfrac{3}{2}(1,0,1,0) + \dfrac{-1/2}{(\frac{1}{2})^2+(\frac{1}{2})^2}\left(\frac{1}{2},0,-\frac{1}{2},0\right) + \dfrac{0}{(\text{who cares})}(\text{whatever})$

$= \left(\frac{3}{2}, 0, \frac{3}{2}, 0\right) - \left(\frac{1}{2}, 0, -\frac{1}{2}, 0\right)$

$= (1,0,2,0)$

Additional space for work on Q4

c) Want v_4 so that $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1/2 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 \\ & v_4 & & \end{bmatrix}$ has rank 4.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1 \\ & v_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ & v_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ & v_4 & & \end{bmatrix}$$

$\therefore v_4 = (0,0,0,1)$ will work.

$\therefore \{(1,0,1,0), (\frac{1}{2},0,-\frac{1}{2},0), (0,1,0,-1), (0,0,0,1)\}$ is a basis for U which extends the one from (a).

5. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 2 & 0 & 2 & 2 \\ -1 & 0 & -1 & -5 \end{bmatrix}.$$

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

(c) Find a basis for $\ker(A) = \{x \in \mathbb{R}^4 \mid Ax = 0\}$.

(d) Find the dot product of each of the vectors in your basis from (a) with each of with the vectors in your basis in (c).

$$A \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) $\text{row}(A) = \text{span}\{(1,0,1,0), (0,0,0,1)\}$
and this spanning set is a basis. (rowspace algorithm)

b) $\text{col}(A) = \text{span}\{(1,2,-1), (-2,2,-5)\}$
and this spanning set is a basis (columnspace algorithm).

c) $[A|0] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -t$
$x_2 = s$
$x_3 = t$
$x_4 = 0$

$\therefore \ker(A) = \{(-t, s, t, 0) \mid s, t \in \mathbb{R}\}$
$= \text{span}\{(-1,0,1,0), (0,1,0,0)\}$

These are basic solutions, so they form a basis of $\ker(A)$

d) since $\ker(A) \perp \text{row}(A)$, all these dot products are zero.

6. (a) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an example which shows that it is false.
- If you say the statement is always true, you must give a clear explanation.

i. If B is an invertible matrix and $AB = 0$ then $A = 0$.

$$AB = 0$$
$$\Rightarrow ABB^{-1} = 0 \cdot B^{-1}$$
$$\Rightarrow AI = 0$$
$$\Rightarrow A = 0$$

ANSWER: True

ii. The rows of a 3×4 matrix are always linearly dependent.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is a 3x4 matrix with L.I. rows.

ANSWER: False

6(b) Let A be a $n \times n$ matrix with real entries. Give three additional statements equivalent to

$$\text{“rank}(A) = n\text{”}$$

i. In terms of the system $Ax = 0$:

Ax=0 has a unique solution (x=0).

ii. In terms of the reduced row echelon form of A:

The RRE form of A is In

or The RRE form of A has n leading 1s.

iii. In terms of the rows of A:

The rows of A are L.I.

or The rows of A span $\mathbb{R}^n$