CONCORDIA UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP 232: MATHEMATICS FOR COMPUTER SCIENCE
FALL 2015

ASSIGNMENT 4: SOLUTIONS

PROBLEM 1.

Use mathematical induction to show that

$$2^n \leq 2^{n+1} - 2^{n-1} - 1,$$

when n is a positive integer.

SOLUTION.

Basis Step. n = 1.

$$\text{LHS} = 2^1 \leq 2^{1+1} - 2^{1-1} - 1 = 2 = \text{RHS}.$$

Inductive Step.

Suppose for some arbitrary n, n > 1,

$$2^n \leq 2^{n+1} - 2^{n-1} - 1.$$

The aim is to show that,

$$2^{n+1} \leq 2^{(n+1)+1} - 2^{(n+1)-1} - 1.$$

To do this, consider the following:

$$\begin{aligned} & 2^n \leq 2^{n+1} - 2^{n-1} - 1 \\ \Rightarrow\ & 2(2^n) \leq 2(2^{n+1} - 2^{n-1} - 1) \\ \Rightarrow\ & 2^{n+1} \leq 2^{n+2} - 2^n - 2 \\ \Rightarrow\ & 2^{n+1} \leq 2^{n+2} - 2^n - 2 + 1 \\ \Rightarrow\ & 2^{n+1} \leq 2^{n+2} - 2^n - 1 \\ \Rightarrow\ & 2^{n+1} \leq 2^{(n+1)+1} - 2^{(n+1)-1} - 1. \end{aligned}$$

Therefore, by the Principle of Mathematical Induction, $2^n \leq 2^{n+1} - 2^{n-1} - 1$, whenever n is a positive integer.

PROBLEM 2.

The sequence of Fibonacci numbers is defined by

$$f_0 = 0, f_1 = 1, \text{ and } f_n = f_{n-1} + f_{n-2}, \text{ for } n > 1.$$

The sequence of Lucas numbers is defined by

$$l_0 = 2, l_1 = 1, \text{ and } l_n = l_{n-1} + l_{n-2}, \text{ for } n > 1.$$

Prove that

$$f_n + f_{n+2} = l_{n+1},$$

whenever n is a positive integer, where f_i and l_i are the ith Fibonacci number and ith Lucas number, respectively.

SOLUTION.

Basis Step. n = 0.

There are two base cases.

$$f_0 + f_2 = 0 + 1 = 1 = l_1,$$

and

$$f_1 + f_3 = 1 + 2 = 3 = l_2,$$

as desired.

Inductive Step. n > 0.

The inductive hypothesis, using strong induction, is

$$f_k + f_{k+2} = l_{k+1}, \text{ for all } k \leq n.$$

Then,

$$\begin{aligned} f_{n+1} + f_{n+3} &= f_n + f_{n-1} + f_{n+2} + f_{n+1} \\ &= (f_n + f_{n+2}) + (f_{n-1} + f_{n+1}) \\ &= l_{n+1} + l_n, \text{ by the inductive hypothesis with } k = n \text{ and } k = n - 1. \\ &= l_{n+2}, \text{ by the definition of the Lucas numbers.} \end{aligned}$$

PROBLEM 3.

For each of the following relations on the set **Z** of integers, determine if it is reflexive, symmetric, anti-symmetric, or transitive. On the basis of these properties, state whether or not it is an equivalence relation or a partial order.

(a) $R = \{(a, b) \mid a^2 = b^2\}$.
(b) $S = \{(a, b) \mid |a - b| \leq 1\}$.

SOLUTION.

(a) R is reflexive, symmetric, not anti-symmetric, transitive, equivalence relation, not a partial order.

(b) S is reflexive, symmetric, not-anti-symmetric, not transitive, not an equivalence relation, not a partial order.

PROBLEM 4.

(a) Prove that $\{(x, y) \mid x - y \in \mathbf{Q}\}$ is an equivalence relation on the set of real numbers, where **Q** denotes the set of rational numbers.
(b) Give [1], [1/2], and [π].

SOLUTION.

(a)

Reflexivity:

$x - x = 0 \in \mathbf{Q}$.

Symmetry:

Let $x - y \in \mathbf{Q}$. Then, $y - x = -(x - y)$ is again a rational number.

Transitivity:

If $x - y \in \mathbf{Q}$ and $y - z \in \mathbf{Q}$, then their sum, namely $x - z$, is also a rational number (as the rational numbers are closed under addition).

(b)

The equivalence class of both 1 and 1/2 is the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number, that is, $\{\pi + r \mid r \in \mathbf{Q}\}$.

PROBLEM 5.

Prove or disprove the following statements:

(a) Let R be a relation on the set $\mathbf{Z}$ of integers such that xRy if and only if $xy \geq 1$. Then, R is irreflexive.
(b) Let R be a relation on the set $\mathbf{Z}$ of integers such that xRy if and only if $x = y + 1$ or $x = y - 1$. Then, R is irreflexive.
(c) Let R and S be reflexive relations on a set A. Then, $R - S$ is irreflexive.

SOLUTION.

(a) R is not irreflexive, as the pair (1, 1) is in the relation.

(b) R is irreflexive, as $n \neq n + 1$ and $n \neq n - 1$, for every integer n. Thus, for every integer n, the pair (n, n) is not in the relation.

(c) $R - S$ is irreflexive. Given that R and S are reflexive, for any element $a \in A$, $(a, a) \in R$ and $(a, a) \in S$. This, in turn, implies that $(a, a) \notin S^c$ and so $(a, a) \notin R \cap S^c$. Now, $R \cap S^c = R - S$. Therefore, $R - S$ is irreflexive.

PROBLEM 6.

Let R be the relation on $\mathbf{Z}^+$ defined by xRy if and only if $x < y$. Then, in the Set Builder Notation, $R = \{(x, y) \mid y - x > 0\}$.

(a) Use the Set Builder Notation to express the transitive closure of R.
(b) Use the Set Builder Notation to express the composite relation R^n, where n is a positive integer.

SOLUTION.

(a) $R^* = R = \{(x, y) \mid y - x > 0\}$.

(b) $R^n = \{(x, y) \mid y - x \geq n\}$.

PROBLEM 7.

(a) Give the transitive closure of the relation $R = \{(a, c), (b, d), (c, a), (d, b), (e, d)\}$ on $\{a, b, c, d, e\}$.
(b) Give an example to show that when the symmetric closure of the reflexive closure of the transitive closure of a relation is formed, the result is not necessarily an equivalence relation.

SOLUTION.

(a) $R^* = \{(a, a), (a, c), (b, b), (b, d), (c, a), (c, c), (d, b), (d, d), (e, b), (e, d)\}$.

(b) Let $R = \{(1, 2), (3, 2)\}$ on the set $\{1, 2, 3\}$. Its transitive closure is itself. The reflexive closure of that is $\{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}$. The symmetric closure of that is $\{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$. The result is not transitive, as, for example, (1, 3) is missing. Therefore, this is not an equivalence relation.