ENGR 213, Section P, Fall Semester 2014, Midterm Test 2, Solutions

Problem 1. The differential equation

$$(1+x^2)y^{''} - 2xy^{'} + 2y = 0$$

has a solution $y_1(x) = x$. Find another solution $y_2(x)$ such that the solution set $\{y_1(x), y_2(x)\}$ is a fundamental set of solutions for the given equation on $(-\infty, \infty)$.
Solution 1 by using the ready formula in the Method of Reduction of the Order. First, standard form to determine $p(x)$:

$$y^{''} - \frac{2x}{1+x^2}y^{'} + \frac{2}{1+x^2}y = 0 \Rightarrow \quad \mathbf{p(x) = -\frac{2x}{1+x^2}}.$$

Second, by using the ready formula with $p(x) = -\frac{2x}{1+x^2}$ and $y_1(x) = x$ we obtain

$$\mathbf{y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx} = x \int \frac{e^{\int \frac{2x}{1+x^2}dx}}{x^2} dx = x \int \frac{e^{\ln(1+x^2)}}{x^2} dx$$

$$= x \int \frac{1+x^2}{x^2} dx = x \int \left(\frac{1}{x^2} + 1\right) dx = x\left(-\frac{1}{x} + x\right) = x^2 - 1$$

Hence,

$$\mathbf{y_1(x) = x, \quad y_2(x) = x^2 - 1.}$$

Now, taking the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix} = 2x^2 - (x^2 - 1) = x^2 + 1 \neq 0 \ (> 0)$$

for all $x \in (-\infty, \infty)$. Hence, **the solution set $\mathbf{y_1}$ and y_2 form a fundamental set of solutions for the given differential equation in $(-\infty, \infty)$.**

Another way to show that y_1 and y_2 form a fundamental set: Suppose now that y_1 and y_2 are linearly dependent on $(-\infty, \infty)$. Then, there are c_1 and c_2 such that at least one of them is nonzero and $c_1 x + c_2(1 - x^2) = 0, \quad x \in (-\infty, \infty)$ and this is possible only when $c_1 = c_2 = 0$. Hence, y_1 and y_2 are linearly independent on the interval $(-\infty, \infty)$ and from here they form a fundamental set of solutions for the given differential equation on $(-\infty, \infty)$.

Solution 2 following the theory. We are looking for a second solution in the form:

$$y_2(x) = y_1(x)u(x), \quad y_1 = x.$$

Represent the differential equation in standard form:

$$y^{''} - \frac{2x}{1+x^2}y^{'} + \frac{2}{1+x^2}y = 0.$$

Plug in the DE $y_2(x) = y_1(x)u(x)$:

$$[y_1^{''}u + 2y_1^{'}u^{'} + y_1u^{''}] - \frac{2x}{1+x^2}[y_1u^{'} + y_1^{'}u] + \frac{2}{1+x^2}y_1u = 0$$

and use the fact that y_1 is a solution to simplify the above differential equation:

$$2y_1^{'}u^{'} + y_1u^{''} - \frac{2x}{1+x^2}y_1u^{'} = 0.$$

Using the explicit form of $y_1(x) = x, y_1^{'}(x) = 1$ we obtain a DE for u:

$$\begin{aligned}
&2u^{'} + xu^{''} - \frac{2x^2}{1+x^2}u^{'} = 0 \quad \Rightarrow \quad xu^{''} + \frac{2}{1+x^2}u^{'} = 0\\
\Rightarrow \quad & u^{'} = v, \quad xv^{'} + \frac{2}{1+x^2}v = 0 \quad \Rightarrow \quad \frac{dv}{v} = -\frac{2}{x(1+x^2)}dx\\
\Rightarrow \quad & \text{partial fractions} \int \frac{dv}{v} = \int \left(-\frac{2}{x} + \frac{2x}{1+x^2}\right) dx\\
\Rightarrow \quad & \ln(v) = -2\ln(x) + \ln(1+x^2) \quad \Rightarrow \quad v = \frac{1+x^2}{x^2} = \frac{1}{x^2} + 1\\
\Rightarrow \quad & u^{'} = \frac{1}{x^2} + 1 \quad \Rightarrow \quad u = \int \left(\frac{1}{x^2} + 1\right) dx \quad \Rightarrow \quad u = -\frac{1}{x} + x\\
\Rightarrow \quad & y_2 = y_1u = x\left(-\frac{1}{x} + x\right) = x^2 - 1.
\end{aligned}$$

From here:

$$\mathbf{y_2 = y_1u = x\left(-\frac{1}{x} + x\right) = x^2 - 1}.$$

Problem 2. Solve the given differential equation

$$y'' + 4y' + 4y = e^{-2x} .$$

(a) By using the method of undetermined coefficients.
(b) By using the method of variation of parameters.

Solution of (a). Find the general solution y_c of the associated homogeneous DE:

$$y'' + 4y' + 4y = 0 \Rightarrow m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m_1 = m_2 = -2.$$
$$\Rightarrow \mathbf{y_c = c_1 e^{-2x} + c_2 x e^{-2x}}.$$

Next, find the form of a particular solution y_p of the given differential equation. Consider the right-hand side is $f(x) = e^{-2x}$ and differentiate it consecutively:

$$f(x) = e^{-2x},\ f^{'}(x) = -2e^{-2x},\ f^{''}(x) = 4e^{-2x}, \ldots$$

to conclude that all derivatives of $f(x)$ (including $f(x)$) have the form

$$Ae^{-2x}, \quad A \quad \text{is a coefficient to be determined}$$

and from here, the preliminary form of the particular solution is

$$y_p(x) = Ae^{-2x}.$$

However, applying undetermined coefficients, we have to avoid the effect of overlapping of $y_p(x)$ with $y_c(x)$. To ensure that there is no overlapping, we must compare $y_c(x)$ to the preliminary form of $y_p(x)$. Since $y_p = Ae^{-2x}$ matches e^{-2x} (a portion of y_c), there is an overlapping of $y_p(x)$ with a part of $y_c(x)$. Thus, you must guess another y_p by multiplying the initial y_p by x, until there is no overlapping with y_c. So, guessing $y_p = Axe^{-2x}$. Yet, there is still an overlapping with a part of y_c. So, guessing $y_p = Ax^2e^{-2x}$, there is no more overlapping, and this is the preliminary form of the particular solution:

$$\mathbf{y_p(x) = Ax^2e^{-2x}} .$$

Plug-in y_p into the DE, $y'' + 4y' + 4y = e^{-2x}$ in order to solve for A. To do this, we must first find $y_p^{'}$ and $y_p^{''}$:

$$y_p = Ax^2e^{-2x} \Rightarrow y_p' = 2Axe^{-2x} - 2Ax^2e^{-2x}$$
$$\Rightarrow y_p'' = 2Ae^{-2x} - 4Axe^{-2x} - 4Axe^{-2x} + 4Ax^2e^{-2x}$$
$$\Rightarrow y_p'' = (2A - 8Ax + 4Ax^2)e^{-2x}$$

Plugging these values into the differential equation in order to solve for the constant A:

$$(2A - 8Ax + 4Ax^2)e^{-2x} + 4(2Ax - 2Ax^2)e^{-2x} + 4(Ax^2e^{-2x}) = e^{-2x}$$

$$\Rightarrow 2Ae^{-2x} = e^{-2x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

Thus, since we have guessed $y_p = Ax^2e^{-2x}$ the particular solution is

$$\mathbf{y_p = \frac{1}{2}x^2e^{-2x}}.$$

Finally, the general solution of the given differential equation is

$$\mathbf{y = y_c + y_p \quad \Rightarrow \quad y = c_1e^{-2x} + c_2xe^{-2x} + \frac{1}{2}x^2e^{-2x}}.$$

Solution of (b). Find the general solution y_c of the associated homogeneous DE:

$$y'' + 4y' + 4y = 0 \Rightarrow m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m_1 = m_2 = -2.$$
$$\Rightarrow \quad y_1 = e^{-2x},\ y_2 = xe^{-2x}$$
$$\Rightarrow \quad \mathbf{y_c = c_1 y_1 + c_2 y_2} \quad \Rightarrow \quad \mathbf{y_c = c_1 e^{-2x} + c_2 x e^{-2x}}.$$

Next, we replace the constants c_1 and c_2 with functions $u_1(x)$ and $u_2(x)$ and we look for y_p in the form:

$$\mathbf{y_p = u_1 y_1 + u_2 y_2} \quad \Rightarrow \quad \mathbf{y_p = u_1 e^{-2x} + u_2 x e^{-2x}}.$$

In other words, we vary the parameters $\mathbf{c_1}$ and $\mathbf{c_2}$, replacing them by functions $\mathbf{u_1}$ and $\mathbf{u_2}$ in order to find $\mathbf{y_p}$.
In order to apply the method of variation of parameters first, we have to put the equation in standard form. However, the given differential equation is in standard form and we proceed by finding u_1' and u_2' with $f(x) = e^{-2x}$:

$$u_1' = \frac{W_1}{W} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & xe^{-2x} \\ e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}} = \frac{-xe^{-4x}}{e^{-4x}} = -x.$$

$$u_2' = \frac{W_2}{W} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix}} = \frac{e^{-4x}}{e^{-4x}} = 1.$$

Then,

$$u_1 = \int -x dx = -\frac{x^2}{2}, \quad u_2 = \int 1 dx = x$$

Since $y_p = u_1 y_1 + u_2 y_2$,

$$y_p = \left[-\frac{x^2}{2}\right]\left[e^{-2x}\right] + [\,x\,]\left[x\,e^{-2x}\right] = \mathbf{\frac{1}{2} x^2 e^{-2x}}.$$

Finally, the general solution of the given differential equation is

$$\mathbf{y = y_c + y_p} \quad \Rightarrow \quad \mathbf{y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2} x^2 e^{-2x}}.$$

Problem 3. Solve the following initial value problem

$$y^{''} + 5x^{-1}y^{'} + 5x^{-2}y = 0, \quad x > 0.$$

$$y(1) = 0, \quad y^{'}(1) = 1.$$

Solution. This is Cauchy-Euler diffrential equation:

$$x^2\, y^{''} + 5x\, y^{'} + 5\, y = 0, \quad x > 0.$$

Looking for a solution in a form $y = x^m$ we obtain the auxiliary quadratic:

$$m(m-1) + 5m + 5 = 0 \quad \Rightarrow \quad m^2 + 4m + 5 = 0 \quad \Rightarrow \quad m_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$\Rightarrow m_1 = -2 - i; \;\; m_2 = -2 + i\,.$$

We have the case of complex conjugate solutions $m_1 = -2 - i$ and $m_2 = -2 + i$. Compex fundamental set of solutions:

$$\tilde{y}_1 = x^{-2-i} = x^{-2}\left[\cos(\ln(x)) \; - \; i\sin(\ln(x))\right],$$

$$\tilde{y}_2 = x^{-2+i} = x^{-2}\left[\cos(\ln(x)) \; + \; i\sin(\ln(x))\right].$$

Real fundamental set of solutions:

$$y_1 = \mathbf{Re}(\tilde{y}_2) = x^{-2}\cos(\ln(x)), \quad y_2 = \mathbf{Im}(\tilde{y}_2) = x^{-2}\sin(\ln(x)).$$

The general solution:

$$\mathbf{y} = \mathbf{c_1 x^{-2}}\cos(\ln(\mathbf{x})) + \mathbf{c_2 x^{-2}}\sin(\ln(\mathbf{x})).$$

$$\mathbf{y}(\mathbf{x}) = \mathbf{x^{-2}}\left[\mathbf{c_1}\,\cos(\ln(\mathbf{x})) + \mathbf{c_2}\,\sin(\ln(\mathbf{x}))\right].$$

Applying the initial value conditions ($\ln(1) = 0$):

$$y(1) = 0 \Rightarrow (1)^{-2}\left[c_1\,\cos(\ln(1)) + c_2\,\sin(\ln(1))\right] \Rightarrow \mathbf{c_1 = 0}$$

$$\Rightarrow y = c_2 x^{-2}\,\sin(\ln(x)).$$

$$y^{'}(x) = c_2(-2x^{-3})\,\sin(\ln(x)) + c_2 x^{-2}[\ln(x)]^{'}\cos(\ln(x))$$

$$= c_2(-2x^{-3})\,\sin(\ln(x)) + c_2 x^{-3}\cos(\ln(x))$$

$$y^{'}(1) = c_2(-2(1)^{-3})\,\sin(\ln(1)) + c_2(1)^{-3}\cos(\ln(1)) = c_2$$

$$\Rightarrow y^{'}(1) = 1 \quad \Rightarrow \quad \mathbf{c_2 = 1}\,.$$

Finally, the unique solution of the given IVP is:

$$\mathbf{y}(\mathbf{x}) = \mathbf{x^{-2}}\,\sin(\ln(\mathbf{x})) \quad \Leftrightarrow \quad \mathbf{y}(\mathbf{x}) = \frac{\sin(\ln(\mathbf{x}))}{\mathbf{x^2}}\,.$$

Problem 4. A mass weighing 16 lb stretches a spring 8 feet. The mass is initially released 2 ft below its equilibrium position with downward velocity 4 ft/sec.
[**5 marks**] **(a)** Formulate an initial value problem (that is: a differential equation and initial value conditions) describing the dynamics of the spring-mass system.
[**10 marks**] **(b)** Find the equation of the spring-mass motion.
[**10 marks**] **(c)** Represent the equation obtained in **(b)** in an amplitude–(phase-angle) form. Point out the following: the amplitude, the phase-angle, the period, and the frequency of the spring-mass motion.
[**5 marks**] **(d)** Find the first time-moment at which the mass passes through the equilibrium position. What is the velocity of the mass at this time-moment – upward or downward?
Assume 32 ft/sec^2 *the acceleration due to gravity.*

Solution. (a)

$$W = F \quad \Rightarrow \quad 16 = k\,8 \quad \Rightarrow \quad \mathbf{k = 2}.$$

$$W = m\,g \quad \Rightarrow \quad 16 = 32\,m \quad \mathbf{m = \frac{1}{2}\ slugs}.$$

The IVP:

$$\mathbf{\frac{1}{2}x''(t) \,+\, 2x(t) = 0 \quad \Leftrightarrow \quad x''(t) \,+\, 4x(t) = 0}$$

$$\mathbf{x(0) = 2}, \quad \mathbf{x'(0) = 4}.$$

(b) The auxiliary quadratic: $m^2 + 4 = 0 \quad \Rightarrow \quad m_{1,2} = \pm\, i\,2$.
Complex fundamental set of solutions (by Euler's formula):

$$\mathbf{\tilde{x}_1(t) = e^{-\,i2\,t}} = \text{(by Euler's formula)}\ \cos(\mathbf{2t}) \,-\, \mathbf{i}\sin(\mathbf{2t}),$$

$$\mathbf{\tilde{x}_2(t) = e^{i2\,t}} = \text{(by Euler's formula)}\ \cos(\mathbf{2t}) \,+\, \mathbf{i}\sin(\mathbf{2t}).$$

Real fundamental set of solutions:

$$\mathbf{x_1(t) = Re}\,[\mathbf{\tilde{x}_2(t)}] = \cos(\mathbf{2t}), \quad \mathbf{x_2(t) = Im}\,[\mathbf{\tilde{x}_2(t)}] = \sin(\mathbf{2t}).$$

The general solution of the differential equation:

$$\mathbf{x(t) \,=\, c_1}\ \cos(\mathbf{2t}) \,+\, \mathbf{c_2}\ \sin(\mathbf{2t}).$$

Applying the initial value conditions:

$$x(0) = 2 \quad \Rightarrow \quad \mathbf{c_1 = 2}.$$

$$x^{'}(t) = -2c_1 \sin(2t) \, + \, 2\, c_2 \, \cos(2t) \quad \Rightarrow \quad x^{'}(0) = 2\, c_2 = 4 \quad \Rightarrow \quad \mathbf{c_2 = 2}.$$

The equation of the given spring-mass motion:

$$\mathbf{x(t) \; = \; 2 \; \cos(2t) \; + \; 2 \; \sin(2t)}.$$

(c)

$$\begin{aligned} x(t) \; &= \; \sqrt{2^2+2^2}\left[\frac{2}{\sqrt{2^2+2^2}} \; \cos(2t) \; + \; \frac{2}{\sqrt{2^2+2^2}} \; \sin(2t)\right] \\ &= 2\sqrt{2}\left[\frac{1}{\sqrt{2}} \; \cos(2t) \; + \; \frac{1}{\sqrt{2}} \; \sin(2t)\right] \\ &= 2\sqrt{2}\left[\sin\left(\frac{\pi}{4}\right) \; \cos(2t) \; + \; \cos\left(\frac{\pi}{4}\right) \; \sin(2t)\right] \\ &= 2\sqrt{2} \; \sin\left(2t \; + \; \frac{\pi}{4}\right). \end{aligned}$$

The amplitude–phase-angle representation of the equation of the spring-mass motion:

$$\mathbf{x(t) \; = \; 2\sqrt{2} \; \sin\left(2t \; + \; \frac{\pi}{4}\right)} \qquad \left(\mathbf{x(t) \; = \; A} \; \sin\left(\omega \, \mathbf{t} \; + \; \mathbf{\Phi}\right)\right).$$

or equivalently

$$\mathbf{x(t) \; = \; 2.828427} \; \sin\left(\mathbf{2t \; + \; 0.7853982}\right)$$

Then:
Amplitude A $= 2\sqrt{2} = 2.828427$,
Phase-angle: $\mathbf{\Phi} = \pi/4 = 0.7853982$,
Period of the motion: $\mathbf{P} = 2\pi/\omega$, $\omega = \sqrt{k/m} = \sqrt{4} = 2 \; rad/sec$, $\mathbf{P} = \pi = 3.141593$,
Frequency of the motion: $\mathbf{F} = 1/P = 1/\pi \, Hz = 0.3183099 \, Hz$.

(d)

$$2t^* + \frac{\pi}{4} = k\pi$$

For $k = 0$ negative time so, $k = 1$ and from here

$$2t^* + \frac{\pi}{4} = \pi \quad \Rightarrow \quad 2t^* = \frac{3\pi}{4} \quad \Rightarrow \quad \mathbf{t^* = \frac{3\pi}{8}}.$$

Graphically, first the motion will attain its max position, next it will attain for the first time its equilibrium position obviously with upward (negative)

velocity. This can be concluded by the amplitude–phase-angle representation of the equation of the spring-mass motion:

$$x^{'}(t) = 4\sqrt{2}\,\cos\left(2t + \frac{\pi}{4}\right) \quad \Rightarrow \quad x^{'}(t^{*}) = x^{'}\left(\frac{3\pi}{8}\right) = 4\sqrt{2}\,\cos\left(\frac{6\pi}{8} + \frac{\pi}{4}\right)$$

$$= 4\sqrt{2}\,\cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = 4\sqrt{2}\,\cos(\pi) \;=\; -\mathbf{4\sqrt{2}}\text{ ft/sec} = \;-\mathbf{5.656854}\text{ ft/sec}$$

hence, **upward velocity $\mathbf{4\sqrt{2} = 5.656854}$ ft/sec.**