MAT 2384

HW #6 solutions

6.6 The series converges if

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{k^{n+1}}x^{2(n+1)}}{\frac{(-1)^n}{k^n}x^{2n}}\right| \quad \text{exists and is } <1,$$

(and diverges if the limit is >1).

The limit simplifies (from cancellation) to

$$\lim_{n\to\infty}\left|\frac{-x^2}{k}\right| = \left|\frac{-x^2}{k}\right| = \frac{|x|^2}{|k|}.$$

no n-dependence!

For convergence, $\frac{|x|^2}{|k|}<1$, i.e. $|x|<|k|^{1/2}$.

(Really, $=$ may also give convergence, but $>$ will give divergent series.)
So the radius of convergence is $|k|^{1/2}$
(See page for the derivative.)

6.8. Again, we look at the limit of the ratio of two successive terms:

$$\lim_{n\to\infty}\left|\frac{\frac{(4(n+1))!}{((n+1)!)^4}x^{n+1}}{\frac{(4n)!}{(n!)^4}x^n}\right|.$$

Note that $(n+1)! = (n+1)\cdot n!$, and

$$(4(n+1))! = (4n+4)! = (4n+4)(4n+3)(4n+2)(4n+1)\,4n!\,,\text{ so}$$

The limit becomes

$$\lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4} x \right| .$$

Divide top and bottom by the highest power of n:

$$= \lim_{n\to\infty} \left| \frac{(4+\frac{4}{n})(4+\frac{3}{n})(4+\frac{2}{n})(4+\frac{1}{n})}{(1+\frac{1}{n})^4} x \right|$$

$$= 4^4 \cdot |x| = 256|x|.$$

For convergence, we need (ignoring the $=1$ case)

$$256\,|x| < 1.$$
$$|x| < \frac{1}{256}$$

So the radius of convergence is $R = \frac{1}{4^4} = \frac{1}{256}$.

(See page for the derivative.)

6.13 Write $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, with corresponding

derivatives: $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$
$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$

Then the D.E. becomes

$$(x^2-1)(2a_2 + 6a_3x + \cdots) \quad + 4x(a_1 + 2a_2x + 3a_3x^2 + \cdots)$$

$$+ 2(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = 0.$$

Then we match up the constant, linear, quadratic coeff's, including extra terms from the $\cdots$ if required.

constant:	$-2a_2 + 2a_0 = 0$	$a_2 = a_0$
linear:	$-6a_3 + 4a_1 + 2a_1 = 0$	$a_3 = a_1$
quadratic:	$2a_2 - 12a_4 + 8a_2 + 2a_2 = 0$	$a_4 = a_2$
	from: $x^2 \cdot y''$ $\quad -y''$ $\quad 4xy'$ $\quad 2y$	
cubic:	$6a_3 - 20a_5 + 12a_3 + 2a_3 = 0$	$a_5 = a_3$
quartic:	$12a_4 - 30a_6 + 16a_4 + 2a_4 = 0.$	$a_6 = a_4.$

So, in order:

$$a_2 = a_0$$
$$a_3 = a_1$$
$$a_4 = a_2 = a_0$$
$$a_5 = a_3 = a_1$$
$$a_6 = a_4 = a_0.$$
$$\vdots$$

[We can easily predict the general answer, $a_{even} = a_0$
$a_{odd} = a_1$.

For proof, the general equation is (for the degree n coeff.)

$$-(n+2)(n+1)\,a_{n+2} \;+\; n(n-1)\,a_n \;+4n\,a_n \;+2a_n \;=0,$$

$$\text{so}\quad a_{n+2} = \frac{n(n-1)+4n+2}{(n+2)(n+1)}\,a_n$$

$$= \frac{n^2+3n+2}{n^2+3n+2}\,a_n \quad = a_n. \qquad]$$

So the general solution is

$$y = a_0 + a_1 x + a_0 x^2 + a_1 x^3 + a_0 x^4 + a_1 x^5 + a_0 x^6 + \cdots$$

$$= a_0\left(1+x^2+x^4+x^6+\cdots\right) \quad + \quad a_1\left(x+x^3+x^5+\cdots\right)$$

$$= a_0\,\frac{1}{1-x^2} \quad + a_1\,\frac{x}{1-x^2}.$$

<u>6.6</u>(b) For the derivative, if

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{k^n}\,x^{2n},$$

then

$$y'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{k^n}\,2n\,x^{2n-1}.$$

(Alternately, we can start the sum at $n=1$, since the $n=0$ term is 0.)

The ratio test gives

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{k^{n+1}}\,2(n+1)\,x^{2(n+1)-1}}{\frac{(-1)^n}{k^n}\,2n\;x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{-(n+1)\,x^2}{k\,n}\right|$$

$$= \left|\frac{x^2}{k}\right|\lim_{n\to\infty}\frac{n+1}{n} = \left|\frac{x^2}{k}\right|\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1} = \left|\frac{x^2}{k}\right|.$$

Again, convergence if $\left|\frac{x^2}{k}\right| < 1$, $\quad |x| < |k|^{1/2}$.

6.8(b) Now $y(x) = \sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}x^n$, $\quad y'(x) = \sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\,n\,x^{n-1}$.

Ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(4(n+1))!}{((n+1)!)^4}\,(n+1)\;x^{(n+1)-1}}{\frac{(4n)!}{(n!)^4}\,n\;x^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\times\frac{(4+\frac{4}{n})(4+\frac{3}{n})(4+\frac{2}{n})(4+\frac{1}{n})}{(1+\frac{1}{n})^4}\right|$$

$$= |x|\lim_{n\to\infty}\left|\frac{(1+\frac{1}{n})(4+\frac{4}{n})(4+\frac{3}{n})(4+\frac{2}{n})(4+\frac{1}{n})}{1\cdot(1+\frac{1}{n})^4}\right| = 256|x|,$$

and again radius of conv. is $\frac{1}{256} = \frac{1}{4^4}$.

6.32 $I = \int_{0.3}^{1.7} e^{-x^2}\,dx = \frac{1.7-0.3}{2}\int_{-1}^{1} \underbrace{e^{-\left(\frac{(1.7-0.3)t+1.7+0.3}{2}\right)^2}}_{f(t)}\,dt.$

see p. 150 in notes.

For the 3-point Gaussian quadrature, we need to evaluate at $t = \pm\sqrt{\frac{3}{5}}, 0$.

At $t = -\sqrt{\frac{3}{5}}$, $f(t) = e^{-(---)^2} = 0.810937$ (to 6 places)

At $t = 0$, $f(t) = e^{-(--)^2} = 0.367879$

At $t = \sqrt{\frac{3}{5}}$, $f(t) = 0.092696$,

Gaussian quadrature is $\frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$

$= 0.829022$ (to 6 places).

So estimate of I is $0.7\cdot(0.829022) = 0.580315$

(Actual value is 0.580624, so pretty good accuracy)

10.4 $x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2$, $f(x,y) = x^2 + y^2$.

$y_0 = 1$.

$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.1(0^2 + 1^2) = 1.1$

$y_2 = y_1 + h f(x_1, y_1) = 1.1 + 0.1(0.1^2 + 1.1^2) = 1.222$

Here $y_0 = y_0^c = 1$.

10.9 $y_1^P = y_0^c + hf(x_0, y_0^c) = 1.1$

$$y_1^c = y_0^c + \tfrac{1}{2}h\left[f(x_0, y_0^c) + f(x_1, y_1^P)\right]$$

$$= 1 + \tfrac{1}{2}(0.1)\left((0^2+1^2) + ((0.1)^2+(1.1)^2)\right)$$

$$= 1.111.$$

$$y_2^P = y_1^c + hf(x_1, y_1^c) = 1.111 + 0.1\left((0.1)^2 + (1.111)^2\right)$$

$$\sim 1.2354 \quad \text{(4 places)}$$

$$y_2^c = y_1^c + \tfrac{1}{2}h\left[f(x_1, y_1^c) + f(x_2, y_2^P)\right]$$

$$= 1.111 + \tfrac{1}{2}(0.1)\left(\left((0.1)^2 + (1.111)^2\right) + \left((0.2)^2 + (1.2354)^2\right)\right)$$

$$\approx 1.2515 \quad \text{(4 places)}$$

10.11 For y_1 calculation, $n=0$.

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1) = 0.1(0.05^2 + 1.05^2) = 0.1105$

$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2) = 0.1(0.05^2 + 1.05525^2)$

≈ 0.111605 (to 6 places)

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1(0.1^2 + 1.111605^2) \sim 0.124567$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$= 1.111463$ (to 6 places).

This turns out to be accurate to 6 places.

For y_2 calc., $n=1$

$$k_1 = hf(x_1, y_1) = 0.1(0.1^2 + 1.11143^2) \approx 0.124535$$
$$k_2 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1) \approx 0.140007$$
$$k_3 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2) \sim 0.141829$$
$$k_4 = hf(x_1 + h, y_1 + k_3) \sim 0.161066$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \sim 1.252976$$

The true value is 1.253017, for an error of -4×10^{-5}.