MAT 2384
HW # 5 solutions

5.24 $F(s) = \dfrac{s^2-5}{s^3+s^2+9s+9}$.

First step is to factor the denominator: $s^3+s^2+9s+9 = (s+1)(s^2+9)$.

(one way to find the $s+1$ factor is to plug in divisors of the constant term 9 into the formula, i.e. $\pm 1, \pm 3, \pm 9$, to see that -1 is a root).

Next, partial fractions:

$$\frac{s^2-5}{(s+1)(s^2+9)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9},$$

then $s^2-5 = A(s^2+9) + (Bs+C)(s+1)$.

Plug in $s=-1$ to get

$$(-1)^2-5 = A((-1)^2+9) + B\cdot 0$$

$$-4 = 10A, \quad A = -\frac{2}{5}.$$

To get B, C, we look at the coefficients of different powers of s:

constant coeff. gives: $-5 = 9A + C$, so

$$C = -9A - 5 = \frac{18-25}{5} = -\frac{7}{5}.$$

linear coeff. gives $0 = B + C$, so $B = -C = \frac{7}{5}$, so

$$\frac{s^2-5}{(s+1)(s^2+9)} = \frac{-2/5}{s+1} + \frac{7/5\, s - 7/5}{s^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2-5}{(s+1)(s^2+9)}\right\} = -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= -\frac{2}{5}e^{-t} + \frac{7}{5}\cos(3t) - \frac{7}{5}\frac{\sin(3t)}{3}.$$

5.32. First write $f(t)$ in terms of shifted Heaviside functions.

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1 & t \geq 1. \end{cases}$$

(Note: $t \geq 1$ is the same as $1 \leq t$.)

$$= t(1 - u(t-1)) + 1(u(t-1))$$

$$= t - (t-1)u(t-1).$$ (collecting multiples of $u(t-1)$)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{(t-1)u(t-1)\}$$

$\left[\begin{array}{l}\text{Write } g(t-1) = t-1, \\ \text{then } g(t) = (t+1) - 1 = t\end{array}\right]$

$$= \frac{1}{s^2} - e^{-1 \cdot s}\mathcal{L}\{t\}$$

replace each occurrence of t with $t+1$.

$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2} = \frac{1}{s^2}(1 - e^{-s}).$$

5.43 $\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin t\}$, $y(0) = 0$, $y'(0) = 0$. Write $Y = \mathcal{L}\{y\}$

So

$$s^2 Y - s \cdot 0 - 0 + Y = \frac{1}{s^2+1}$$

$$(s^2+1)Y = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)^2}.$$ How do we find $y = \mathcal{L}^{-1}\{Y\}$?

Partial fractions won't help, it's already in simple form

$\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}\right.$ would be the general form, but here $A = B = C = 0$, $\left.D = 1\right)$.

Since $Y(s)$ is a product, we can use convolutions!

$$Y(s) = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1} = F(s) \cdot F(s) \quad \text{where } F(s) = \frac{1}{s^2+1}$$

Then $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{F(s) \cdot F(s)\}$

$$= \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{F(s)\}$$

$$= \sin t * \sin t$$

$$= \int_0^t (\sin \tau)(\sin(t-\tau))\, d\tau$$

This is a slightly complicated integral, but not impossible. First use the addition formula for sin:

$$\sin(t-\tau) = \sin t \cos(-\tau) + \cos t \sin(-\tau)$$

$$= \sin t \cos \tau - \cos t \sin \tau,$$

Then plug into the integral:

$$\int_0^t (\sin \tau)(\sin t \cos \tau - \cos t \sin \tau)\, d\tau$$

$$= \sin t \left(\int_0^t \sin \tau \cos \tau \, d\tau \right) - \cos t \int_0^t \sin^2 \tau \, d\tau.$$

remember, t is a constant for the τ integral

$$= \sin t \left(\tfrac{1}{2} \sin^2 \tau \Big|_0^t \right) - \cos t \left(\int_0^t \frac{1-\cos 2\tau}{2}\, d\tau \right)$$

you will be provided with this identity if required.

$$= \sin t \left(\tfrac{1}{2} \sin^2 t \right) - \cos t \left(\left(\frac{\tau}{2} - \frac{\sin 2\tau}{4} \right)\Big|_0^t \right)$$

$$= \tfrac{1}{2} \sin^3 t - \cos t \left(\frac{t}{2} - \frac{\sin 2t}{4} \right)$$

This does not look like the answer we'd have using undetermined coeff's, but it is the same function nonetheless:

$$y(t) = \frac{1}{2}\sin^3 t - \cos t\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)$$

$$= \frac{1}{2}\sin^3 t - \cos t\left(\frac{t}{2} - \frac{2\sin t\cos t}{4}\right)$$

$$= \frac{1}{2}\sin^3 t - \frac{t}{2}\cos t + \frac{1}{2}\sin t\cos^2 t$$

$$= -\frac{t}{2}\cos t + \frac{1}{2}\sin t\,(\underbrace{\sin^2 t + \cos^2 t}_{=1})$$

$$= -\frac{t}{2}\cos t + \frac{1}{2}\sin t.$$

(You do not need to do this reduction.)

5.54 $\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{u(t-1) + \delta(t-2)\}$, $y(0) = 0$, $y'(0) = 1$

(Write $Y = \mathcal{L}\{y\}$)

$$s^2Y - s\cdot 0 - 1 + 5(sY - 0) + 6Y = \frac{e^{-s}}{s} + e^{-2s}$$

$$(s^2 + 5s + 6)Y = 1 + \frac{e^{-s}}{s} + e^{-2s}.$$

$$Y = \frac{\left(1 + \frac{e^{-s}}{s} + e^{-2s}\right)}{s^2 + 5s + 6} = \frac{1 + e^{-2s}}{s^2 + 5s + 6} + \frac{e^{-s}}{s(s^2 + 5s + 6)}.$$

(Note that the exponentials won't contribute to the partial fractions, but the $\frac{1}{s}$ will.)

Now we need to do partial fractions:

$$\frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$

(I skipped the steps; you may make use of the fact $\frac{1}{b(b+1)} = \frac{1}{b} - \frac{1}{b+1}$.)

$$\frac{1}{s(s^2+5s+6)} = \frac{1}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3},$$

$$1 = (s+2)(s+3)A + s(s+3)B + s(s+2)C.$$

Plug in $s=0$ to get $1=6A$, $A=\frac{1}{6}$
Plug in $s=-2$ to get $1=-2B$, $B=-\frac{1}{2}$.
Plug in $s=-3$ to get $1=3C$, $C=\frac{1}{3}$.

So $\frac{1}{s(s^2+5s+6)} = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}$.

So
$$Y(s) = (1+e^{-2s})\left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-s}\left(\frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}\right),$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\} \quad (1)$$
$$+ \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right)\right\} \quad (2)$$
$$+ \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}\right)\right\} \quad (3)$$

(separating the different exponential terms.

We have three inverse Laplace transforms to compute.

The first is easy: ① $= e^{-2t} - e^{-3t}$.

For the second, write it as $\mathcal{L}^{-1}\{e^{-2s}F(s)\}$, then

$F(s) = \frac{1}{s+2} - \frac{1}{s+3}$, $\quad f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-2t} - e^{-3t}$, and

$$\mathcal{L}^{-1}\{e^{-2s}F(s)\} = u(t-2)\,f(t-2)$$

(replace t with t-2 in formula for f)

$$= u(t-2)\left(e^{-2(t-2)} - e^{-3(t-2)}\right)$$

$$= u(t-2)\left(e^{4}e^{-2t} - e^{6}e^{-3t}\right)$$

For the third, write it as $\mathcal{L}^{-1}\{e^{-s}F(s)\}$, then

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}$, and

$$\mathcal{L}^{-1}\{e^{-s}F(s)\} = u(t-1)\,f(t-1)$$

$$= u(t-1)\left(\frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)}\right)$$

$$= u(t-1)\left(\frac{1}{6} - \frac{e^{2}}{2}e^{-2t} + \frac{e^{3}}{3}e^{-3t}\right)$$

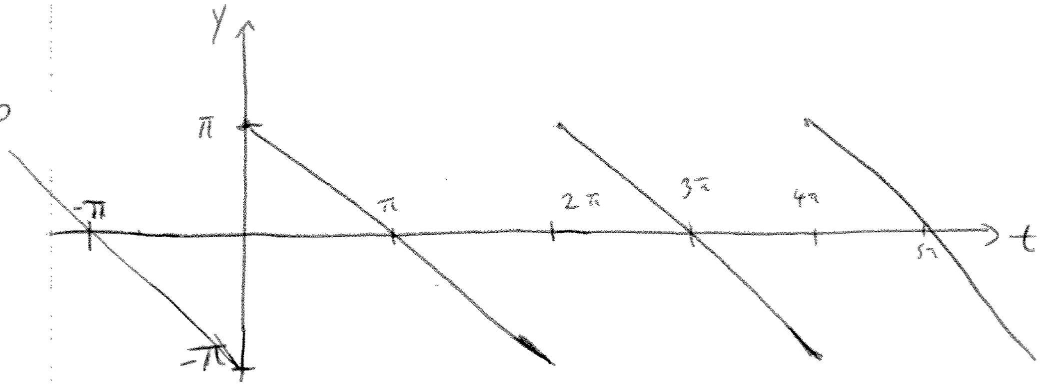
Putting it all together,

$$y(t) = \left(e^{-2t} - e^{-3t}\right) + u(t-1)\left(\frac{1}{6} - \frac{e^{2}}{2}e^{-2t} + \frac{e^{3}}{3}e^{-3t}\right)$$
$$+ u(t-2)\left(e^{4}e^{-2t} - e^{6}e^{-3t}\right).$$

$$= \frac{1}{6}u(t-1) + e^{-2t}\left(1 - \frac{e^{2}}{2}u(t-1) + e^{4}u(t-2)\right)$$
$$+ e^{-3t}\left(-1 + \frac{e^{3}}{3}u(t-1) - e^{6}u(t-2)\right).$$

(either form is OK).

5.60



The function is given by $\pi - t$ on the interval $(0, 2\pi)$, then repeats periodically.

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

period $= 2\pi$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}(\pi - t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{\pi e^{-st}}{-s} \Bigg|_{t=0}^{2\pi} - \int_0^{2\pi} t e^{-st}\,dt \right)$$

parts

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{\pi}{s} - \frac{\pi e^{-2\pi s}}{s} - \left(t \frac{e^{-st}}{-s} \Bigg|_0^{2\pi} \right) + \int_0^{2\pi} \frac{e^{-st}}{-s}\,dt \right)$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{\pi}{s}\left(1 - e^{-2\pi s}\right) + \frac{2\pi e^{-2\pi s}}{s} + \frac{e^{-st}}{(-s)^2} \Bigg|_0^{2\pi} \right)$$

$$= \frac{\pi}{s} + \frac{1}{1-e^{-2\pi s}} \left(\frac{2\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{1}{s^2} \right)$$

9.3 $n=4$, $a=0$, $b=1$, so $h=\frac{1-0}{4}=\frac{1}{4}$, $x_0=0$, $x_1=\frac{1}{4}$, $x_2=\frac{1}{2}$, $x_3=\frac{3}{4}$, $x_4=1$.

Trapezoid rule is

$$\frac{1}{2}h\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right)$$

$$=\frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{1+0}+2\,\frac{1}{1+\frac{1}{4}}+2\,\frac{1}{1+\frac{1}{2}}+2\,\frac{1}{1+3/4}+\frac{1}{1+1}\right)$$

$$=\frac{1171}{1680}\sim 0.69702 \quad \text{(to 5 decimals)}$$

9.4 Simpson's rule is

$$\frac{1}{3}h\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right)$$

$$=\frac{1}{3}\left(\frac{1}{4}\right)\left(\frac{1}{1+0}+4\,\frac{1}{1+1/4}+2\,\frac{1}{1+1/2}+4\,\frac{1}{1+3/4}+\frac{1}{1+1}\right)$$

$$=\frac{1747}{2520}\sim 0.69325 \quad \text{(to 5 decimals)}$$

Actual integral is $\int_0^1 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^1 = \ln(2)\sim 0.69315$,

so Simpson's is an order of magnitude more accurate.)

9.9. Error with left/right is $\frac{M(b-a)h}{2}$, where $M=\max\limits_{a\le x\le b}|f'(x)|$.

Here $f'(x)=\frac{1}{x}$ is largest (on interval $[1,3]$) at $x=1$, with value 1.

So maximum possible error with left/right Riemann sum is $\frac{1(3-1)\cdot h}{2}=h=\frac{b-a}{n}=\frac{2}{n}$.

For error to be at most 10^{-6}, $\frac{2}{n}\le 10^{-6}$, $n\ge\frac{2}{10^{-6}}=2\times10^{6}$

So we'd take $n = 2\times 10^{+6}$, with $h = 10^{-6}$.

Error with trapezoid is $\frac{M}{12}(b-a)h^2$, with $M = \max_{x\in[a,b]} |f''(x)|$.

$|f''(x)| = \left|-\frac{1}{x^2}\right| = \frac{1}{x^2}$ is largest (on interval $[1,3]$) at $x=1$, with value 1, so error

$$= \frac{1}{12}(2)h^2 = \frac{h^2}{6}$$

For error $\leq 10^{-6}$, $\frac{h^2}{6} \leq 10^{-6}$

$$h^2 \leq 6\times 10^{-6}$$
$$h \leq \sqrt{6}\times 10^{-3}.$$

$h = \frac{b-a}{n}$, so $n = \frac{b-a}{h} = \frac{2}{h}$, so we need

$$n \geq \frac{2}{\sqrt{6}\times 10^{-3}} \approx 816.497.$$

The smallest n that works is $n=817$, so that $h = \frac{2}{n} \approx 0.0024480$.

Error with midpoint rule is $\frac{M}{24}(b-a)h^2 = \frac{M}{24}\left(\frac{(b-a)^3}{n^2}\right) = \frac{M}{3}\frac{1}{n^2}$,

with $M = \max_{x\in[1,3]} |f''(x)| = 1$ again.

So we need $\frac{1}{3}\frac{1}{n^2} \leq 10^{-6}$

$$n \geq \sqrt{\frac{1}{3}\times 10^{6}} \sim 577.350$$

The smallest n that works is $n=578$, so that $h = \frac{2}{n} \approx 0.00346021$

Error with Simpsons is $\frac{M}{180}(b-a)h^4 = \frac{M}{180}\frac{(b-a)^5}{n^4} = \frac{2^5 M}{180}\frac{1}{n^4}$, where

$M = \max_{x\in[1,3]} |f^{(4)}(x)|$.

Here $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$.

$|f^{(4)}(x)| = 6x^{-4}$. On the interval $[1, 3]$, this is largest at $x=1$, with value 6.

So maximum possible error is

$$\frac{2^5 \cdot 6}{180} \frac{1}{n^4},$$

For this to be at most 10^{-6}, need

$$n \geq \left(\frac{2^5 \cdot 6}{180} \cdot 10^6 \right)^{1/4} \sim 32.137$$

We need n to be an even number for Simpson's rule, so the smallest n that works is $n = 34$, so that $h = \frac{2}{n} \sim 0.05882$.