MAT 2384

Assignment 4 solutions

(1) 4.9 First we solve the homogeneous DE $\vec{y}\,' = A\vec{y}$, by finding the e-val's and e-vectors of A.

e-val's of A are the roots of

$$\det(A - \lambda I) = 0$$

$$\cancel{\det} \quad (2-\lambda)(-2-\lambda) - (-1)(3) = 0$$

$$\lambda^2 - 1 = 0, \qquad \lambda = \pm 1.$$

E-vec for $\lambda = 1$: $(A - \lambda I)\vec{v} = \vec{0}$.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Write $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, take $v_2 = 1$, and get $v_1 = 1$, so $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

E-vec. for $\lambda = -1$: $A - \lambda I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$

Write $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then $3v_1 - v_2 = 0$. Take $v_1 = 1$ (for convenience), get $v_2 = 3$, so $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Gen'l sol'n of homog. is

$$\vec{y}_h = c_1 e^{x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-x} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

To find a solution to the non-homog. problem, we use undetermined coeffs. (using variation of parameters is OK too.)

$$\vec{r}(x) = \begin{bmatrix} e^x \\ -e^x \end{bmatrix}$$

The functions that appear in $\vec{r}(x)$ are e^x and e^{-x}.

The list of a basis for these functions and their derivatives is just e^x.

Since e^x appears in the formula for $\vec{y}_h$, we need to make a second list:

$$x e^x \qquad , \qquad e^x$$

$\nwarrow$ to go beyond the e^x in $\vec{y}_h$

$\nwarrow$ lower powers need to be included with matrix version of undetermined coeffs.

So $\vec{y}_p = \begin{bmatrix} axe^x + be^x \\ cxe^x + de^x \end{bmatrix}$, some a, b, c, d.

We need $\vec{y}_p\,' = A\vec{y}_p + \vec{r}(x)$.

$$\text{LHS} = \vec{y}_p\,' = \begin{bmatrix} axe^x + (a+b)e^x \\ cxe^x + (c+d)e^x \end{bmatrix}.$$

$$\text{RHS} = A\vec{y}_p + \vec{r}(x) = \begin{bmatrix} 2(axe^x + be^x) - (cxe^x + de^x) + e^x \\ 3(axe^x + be^x) - 2(cxe^x + de^x) - e^x \end{bmatrix}$$

$$= \begin{bmatrix} (2a-c)xe^x + (2b-d+1)e^x \\ (3a-2c)xe^x + (3b-2d-1)e^x \end{bmatrix},$$

where we collected e^x terms and xe^x terms.

Matching the xe^x and e^x terms in the LHS and RHS gives

$$\left.\begin{aligned} a &= 2a - c \\ a+b &= 2b - d + 1 \end{aligned}\right\} \text{ first entry}$$

$$\left.\begin{aligned} c &= 3a - 2c \\ c+d &= 3b - 2d - 1 \end{aligned}\right\} \text{ second entry.}$$

We rewrite as 4 equations in 4 unknowns.

$$\begin{aligned} a - c &= 0 && (1) \quad (\text{from } a = 2a - c) \\ a - b + d &= 1 && (2) \\ 3a - 3c &= 0 && (3) \\ c - 3b + 3d &= -1. && (4) \end{aligned}$$

(3) is equivalent to (1), can ignore. (1) says $a = c$, so can replace all c's with a's. This gives

$$\begin{aligned} a - b + d &= 1 && (2) \\ a - 3b + 3d &= -1 && (4'). \end{aligned}$$

$3 \cdot (2) - (4')$ gives $2a = 4$, $a = 2$, so $c = 2$.

Equation (2) then gives $b = -1 + a + d = d + 1$. Picking $d = 0$ gives $b = 1$.
(If we row-reduced, then the fourth column would have no leading entry, and we'd let d be arbitrary.)

Picking another value of d gives another possibility for $\vec{y}_p$.

So $\vec{y}_p = \begin{bmatrix} 2xe^x + e^x \\ 2xe^x \end{bmatrix}$

(Verify: $\vec{y}_p{}' = \begin{bmatrix} 2xe^x + 3e^x \\ 2xe^x + 2e^x \end{bmatrix}$.

$$A\vec{y}_p + \vec{r}(x) = \begin{bmatrix} 2(2xe^x+e^x) - 2xe^x \\ 3(2xe^x+e^x) - 4xe^x \end{bmatrix} + \begin{bmatrix} e^x \\ -e^x \end{bmatrix}$$

$$= \begin{bmatrix} 2xe^x + 3e^x \\ 2xe^x + 2e^x \end{bmatrix} = \vec{y}_p{}'.)$$

The general solution is $\vec{y} = \vec{y}_p + \vec{y}_h$.

(2) 4.11 e-val's: $(5-\lambda)(1-\lambda) + 3 = 0$. $\lambda^2 - 6\lambda + 8 = 0$, $\lambda = 2, 4$.

E-vecs: $\lambda=2$: $\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \sim \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (or $\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$)

$\lambda=4$: $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \sim \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The gen'l sol'n is $\vec{y}_h = c_1 e^{2x} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{4x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then $\vec{y}_h(0) = c_1 e^0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 3c_1 + c_2 \end{bmatrix}$.

We need $\vec{y}_h(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, so

$$c_1 + c_2 = 2 \quad ①$$
$$3c_1 + c_2 = -1 \quad ②$$

① − ② gives $-2c_1 = 3$, $c_1 = -\frac{3}{2}$. Then $c_2 = 2 - c_1 = \frac{7}{2}$.

So the solution to the IVP is

$$\vec{y} = -\frac{3}{2}e^{2x}\begin{bmatrix}1\\3\end{bmatrix} + \frac{7}{2}e^{4x}\begin{bmatrix}1\\1\end{bmatrix}.$$

3) $A = \begin{bmatrix}4 & 1\\-1 & 2\end{bmatrix}$, e-vals: $(4-\lambda)(2-\lambda)+1=0$. $\lambda^2-6\lambda+9=0$, $\lambda=3$. double

E-vecs: $A-\lambda I = \begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}-1\\1\end{bmatrix}$.

We dont have enough e-vec's, so need generalized.

Now solve $(A-\lambda I)\vec{u} = \vec{v}$ with $\vec{v} = \begin{bmatrix}-1\\1\end{bmatrix}$ an e-vec.

$$\left[\begin{array}{cc|c}1 & 1 & -1\\-1 & -1 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & -1\\0 & 0 & 0\end{array}\right].$$

Writing $\vec{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$, have $u_1+u_2=-1$.

Pick $u_2=0$ (or anything else), get $u_1=-1$, so

$\vec{u} = \begin{bmatrix}-1\\0\end{bmatrix}$.

Gen'l sol'n is

$$\vec{y} = c_1e^{3x}\vec{v} + c_2e^{3x}(x\vec{v}+\vec{u})$$
$$= c_1e^{3x}\begin{bmatrix}-1\\1\end{bmatrix} + c_2e^{3x}\left(x\begin{bmatrix}-1\\1\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix}\right).$$

4) 5.1: $\mathcal{L}\{-3t+2\} = -3\mathcal{L}\{t\} + 2\mathcal{L}\{1\}$

$$= -3\frac{1}{s^2} + 2\cdot\frac{1}{s} = -\frac{3}{s^2} + \frac{2}{s} = \frac{2s-3}{s^2}.$$

5) 5.15 $\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s^2-16}\right\}$.

We use partial fractions (in this case, could also use $\cosh(4t)$ and $\sinh(4t)$ but you aren't responsible for those functions).

$s^2-16 = (s+4)(s-4)$,

$$\frac{4(s+1)}{s^2-16} = \frac{A}{s+4} + \frac{B}{s-4}.$$

$$\therefore \quad 4s+4 = A(s-4) + B(s+4)$$

Plug in $s=4$, get

$$20 = \cancel{A(0)} + 8B, \qquad B = \frac{20}{8} = \frac{5}{2}.$$

Plug in $s=-4$, get

$$-12 = A(-8) + \cancel{B(0)}, \qquad A = \frac{12}{8} = \frac{3}{2}.$$

So $\mathcal{L}^{-1}\left\{\frac{4(s+1)}{s^2-16}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\frac{1}{s+4} + \frac{5}{2}\frac{1}{s-4}\right\}$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= \frac{3}{2}e^{-4t} + \frac{5}{2}e^{4t} \qquad \left(\text{since } \mathcal{L}\{e^{at}\} = \frac{1}{s-a}\right)$$

6) (a) $f'(1) \sim \dfrac{f(1+h) - f(1-h)}{2h} = \dfrac{10^{1.05} - 10^{0.95}}{0.1}$

~ 23.076752 to 6 decimal places.

(The actual value is $10^1 \ln(10) \sim 23.025851$, to 6 decimal places)

(b) The error

$$E = \frac{\tilde{f}(1+h) - \tilde{f}(1-h)}{2h} - f'(1),$$

where $\tilde{f}$ means calculator output, is equal to

$$E = \frac{1}{2h}\left[\left(\tilde{f}(1+h) - f(1+h)\right) - \left(\tilde{f}(1-h) - f(1-h)\right)\right]$$

$$+ \frac{f(1+h) - f(1-h)}{2h} - f'(1)$$

Therefore

absolute values should be added!

$$|E| \leq \frac{1}{2h}\left(|\tilde{f}(1+h) - f(1+h)| + |\tilde{f}(1-h) - f(1-h)|\right)$$

$$+ \left| \frac{f(1+h) - f(1-h)}{2h} - f'(1) \right|.$$

The first terms are the roundoff errors, bounded by 5×10^{-10}.

The last term equals $\dfrac{h^2}{6} f'''(\xi)$, for some $\xi \in [1-h, 1+h]$, by equation (9.3) in the text.

So

$$|E| \leq \frac{1}{2h}\left(5\times10^{-10} + 5\times10^{-10}\right) + \frac{h^2}{6}\max_{x\in[1-h,1+h]} f'''(x).$$

Here, $f(x) = 10^x$, $f'(x) = 10^x \ln 10$, $f''(x) = 10^x(\ln 10)^2$,

$f'''(x) = 10^x(\ln 10)^3$.

Among all $x\in[1-h,1+h]$, the largest $f'''(x)$ value occurs at $x = 1+h$, so if $h \leq 0.1$,

$$f'''(x) \leq f'''(1+h) \leq f'''(1.1) = 10^{1.1}(\ln 10)^3 < 153.691$$

for $x\in[1-h, 1+h]$.

Therefore,

$$|E| \leq \frac{5\times10^{-10}}{h} + \frac{153.691}{6}h^2.$$

The function $\frac{5\times10^{-10}}{h} + \frac{153.691}{6}h^2$ is minimized when its derivative $= 0$, ie.

$$-(5\times10^{-10})h^{-2} + 2\,\frac{153.691}{6}h = 0,$$

$$h^3 = \frac{(5\times10^{-10})\,3}{153.691} \sim 9.76\times10^{-12},$$

$$h \sim 2.137 \times 10^{-4}.$$

This value of h should give the most accurate estimate for $f'(1)$ with formula (9.3).