MAT 2384

HW #2 Solutions

2.16 $x^2y'' + xy' + 4y = 0$ has char. eq'n.

$m(m-1) + m + 4 = 0$, i.e. $m^2 + 4 = 0$,

with roots $m = \pm 2i$

So the general solution is $y = c_1\cos(2\ln x) + c_2\sin(2\ln x)$

3.4 $y^{(4)} + y''' - 3y'' - y' + 2y = 0$. Char. eq'n is $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0$

We first plug in some numbers to find roots of char. eq'n.

With $\lambda = 1$, we get $1+1-3-1+2 = 0$ ✓, so $\lambda - 1$ is a factor.

with $\lambda = -1$, we get $1-1-3+1+2 = 0$ ✓, so $\lambda + 1$ is a factor.

Let's do long division, using the known factor $(\lambda-1)(\lambda+1) = \lambda^2 - 1$:

$$\begin{array}{r|l|l}
\lambda^2 - 1 & \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 & \\
 & \lambda^4 \quad - \lambda^2 & \lambda^2 \\ \hline
 & \lambda^3 - 2\lambda^2 - \lambda + 2 & \\
 & \lambda^3 \quad - \lambda & \lambda \\ \hline
 & -2\lambda^2 \quad + 2 & \\
 & -2\lambda^2 \quad + 2 & -2 \\ \hline
 & 0 &
\end{array}$$

we expect a remainder of 0, since $\lambda^2 - 1$ is a factor.

So $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = (\lambda^2 - 1)(\lambda^2 + \lambda - 2) = [(\lambda+1)(\lambda-1)][(\lambda+2)(\lambda-1)]$, factoring the two quadratics. So $\lambda = 1$ is a double root, and $\lambda = -2, -1$ are single roots.

The general solution is then $y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-2x} + c_4 e^{-x}$.

3.15 $W(x^{1/3}, x^{1/4}) = \det \begin{bmatrix} x^{1/3} & x^{1/4} \\ \frac{1}{3}x^{-2/3} & \frac{1}{4}x^{-3/4} \end{bmatrix} = x^{1/3}\left(\frac{1}{4}x^{-3/4}\right) - x^{1/4}\left(\frac{1}{3}x^{-2/3}\right)$

$$= \left(\frac{1}{4} - \frac{1}{3}\right) x^{-5/12} = -\frac{1}{12} x^{-5/12}.$$

This is not the zero function, so $x^{1/2}$ and $x^{1/4}$ are linearly independent.

3.19 We have a solution $y_1(x)$ of a homogeneous linear DE. Then for any constant c_1, the function $c_1 y_1(x)$ is also a solution.

Take, for example, $c = 3$. Then $y = 3e^{x}$ is a solution of the DE, with $y(0) = 3$.

In assignment 3, you'll be asked to find a solution y_2 independent of y_1.

3.23 The homog. DE is $y'' + y' = 0$, with char eq'n $\lambda^2 + \lambda = 0$, roots $\lambda = 0, -1$, and general solution $y = c_1 + c_2 e^{-x}$.

The list of functions in $r(x)$ and its derivatives is:

$x^2, x, 1$.

However, the function 1 is already a solution of the homog. DE, so we must multiply by the next higher power of x, i.e. x. This gives the list

$$x^3, x^2, x,$$

so

$$y_p = c_1 x^3 + c_2 x^2 + c_3 x.$$

Then $y_p' = 3c_1 x^2 + 2c_2 x + c_3$

$$y_p'' = 6c_1 x + 2c_2, \qquad \text{so}$$

$$y_p'' + y_p' = (6c_1 x + 2c_2) + (3c_1 x^2 + 2c_2 x + c_3)$$

$$= 3c_1 x^2 + (6c_1 + 2c_2)x + (2c_2 + c_3).$$

collecting like powers of x

$$= 3x^2 \quad \text{(the RHS of the DE)}$$

Matching coefficients,

$$3c_1 = 3 \qquad (x^2 \text{ coeff.})$$
$$6c_1 + 2c_2 = 0 \qquad (x \text{ coeff.})$$
$$2c_2 + c_3 = 0 \qquad (1 \text{ coeff})$$

these are the functions in our first list.

So $c_1 = 1$, $c_2 = -\frac{6}{2}c_1 = -3$,
$c_3 = -2c_2 = 6$, and
$y_p = x^3 - 3x^2 + 6x.$

The general solution is $y = y_p + y_h$

$$= x^3 - 3x^2 + 6x + c_1 + c_2 e^{-x},$$

c_1, c_2 any constants

7.12 $x_0 = 1, \quad x_1 = 0.5$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 0.5 - \frac{0.5 - 1}{(1 - \tan(0.5)) - (2 - \tan(1))}(1 - \tan(0.5))$$

≈ 20.9271912244

~~[illegible]~~ (to 10 decimal places)

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \sim 0.7959030122$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) \sim 0.5293048222$$

$x_5 \sim -0.7731452615$

$x_6 \sim -0.0614991192$

$x_7 \sim 0.024362437$

$x_8 \sim -0.0000185973$

$x_9 \sim 0.0000000037$

$x_{10} \sim 0$ (to 10 decimals)

So the first time we have agreement to 6 decimal places is with x_9 and x_{10}, and the ~~fixed point~~ root is 0, to 6 decimal places.

We can verify that $p=0$ is exactly a root of f: $f(0) = 2\cdot 0 - \tan(0) = 0$.

For the ~~...~~ order of convergence, we look at the errors, $\varepsilon_i = x_i - p = x_i$

$\frac{\varepsilon_i}{\varepsilon_{i-1}}$ gives the list

$2 = \frac{0.5}{1}$, 20.9, 0.0380, 0.665, −1.46,

0.0795, −0.396, −0.000763,

−0.000199, (0)

(to 3 significant digits in each case, except the last, where we have no ~~...~~ significant digits.)

This looks like it may be converging to 0.

$\frac{\varepsilon_i}{(\varepsilon_{i-1})^2}$ gives 4, 20.9, 0.00182, 0.836, −2.760, −0.103,

6.44, −0.0313, 10.7, (0)

This bounces ~~...~~ around a lot, but doesn't look like it's converging to 0 or ∞.

So we guess that the order is about 2.

8.2 For degree 1, we use $x_0 = 8.3$, $x_1 = 8.6$, the points nearest 8.4.

$$L_0(x) = \frac{x-8.6}{8.3-8.6}, \quad L_1(x) = \frac{x-8.3}{8.6-8.3}$$

$$p_1(x) = 17.56492\,L_0(x) + 18.50515\,L_1(x).$$

$$\text{so } p_1(8.4) = 17.56492\left(\frac{8.4-8.6}{8.3-8.6}\right) + 18.50515\left(\frac{8.4-8.3}{8.6-8.3}\right)$$

$$\approx 11.70995 + 6.16838$$

$$\doteq 17.87833$$ (keeping 5 decimal places throughout).

For degree 3, we use all the points, $x_0 = 8.1$, $x_1 = 8.3$, $x_2 = 8.6$, $x_3 = 8.7$

$$L_0(x) = \frac{(x-8.3)(x-8.6)(x-8.7)}{(8.1-8.3)(8.1-8.6)(8.1-8.7)}, \quad L_1(x) = \cdots$$

$$p_3(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$\text{so } p_3(8.4) = y_0 L_0(8.4) + y_1 L_1(8.4) + y_2 L_2(8.4) + y_3 L_3(8.4)$$

$$\approx 16.94410(-0.1) + 17.56492(0.75)$$

$$+ 18.50515(0.6) + 18.82091(-0.25)$$

$$= 17.87715$$ (rounded up; 5 decimal place accuracy)

(note that $L_0(8.4) + L_1(8.4) + L_2(8.4) + L_3(8.4) = 1$, this is an algebraic property of the Lagrange interpolating polynomials, that they add up to 1.)

8.5 The interpolating poly. of degree three that fits the data points x_0 through x_3 can be read off from the numbers provided:

$$p_3(x) = 22.0 + 8.400(x-3.2) + 2.856(x-3.2)(x-2.7)$$

$$- 0.528(x-3.2)(x-2.7)(x-1.0).$$

Notice that the numbers x_0, x_1, x_2, x_3 did not have to be in increasing order; any order is valid.

The missing entries are (to 3 decimal place accuracy)

$\boxed{2.118}$

$\boxed{2.012}$

$\boxed{6.342}$ $\boxed{-4.273}$

$\boxed{-10.381}$

$\boxed{-41.413}$

Also, $f[x_0, x_1, x_2, x_3, x_4] = -1.561$.

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#9. $y' = e^{-x} + e^{y}$, so $f(x,y) = e^{-x} + e^{y}$.

IC is $y(1) = 5$, so $x_0 = 1$, $y_0 = 5$.

We're given $a = 2$, $b = 1$, so the rectangle is

$$R: \ |x-1| < 2, \quad |y-5| < 1,$$

i.e. $1-2 < x < 1+2$, $5-1 < y < 5+1$

i.e. $-1 < x < 3$, $4 < y < 6$.

The largest $|f(x,y)| = |e^{-x} + e^{y}| = e^{-x} + e^{y}$ can be on this rectangle

exponentials are always positive

is its value when x is as small as possible and y as large, i.e.
at $(-1, 6)$, where the value is $e^{-(-1)} + e^{6} = e + e^{6} \sim 406.1$
So we can take $k = e + e^{6}$.
~~We need an upper bound. Let's take k =~~

By Theorem 3.1, the DE + IC has exactly one solution for x with

$$|x - x_0| < \alpha, \qquad \alpha = \min(a, b/k)$$
$$= \min(2, 1/(e+e^{6})) = \frac{1}{e+e^{6}} \sim 0.002462$$

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The endpoints of this interval are

$$x_0 - \delta = 1 - \frac{1}{e+e^6} \sim 0.99734 \qquad \text{(5 decimal places)}$$

and

$$x_0 + \delta = 1 + \frac{1}{e+e^6} \sim 1.00246.$$

10 (bonus) 2.11 The char. eq'n is $\lambda^2 + 2\lambda + 1 = 0$, with double root $\lambda = -1$ (that's what it means to be critically damped; that there is a negative double root).

So the general solution is $y = c_1 e^{-x} + c_2 x e^{-x}$.

For the IC, we set $1 = y(0) = c_1 e^{-0} + c_2 (0) e^{-0} = c_1$, so $c_1 = 1$

$$1 = y'(0) = -c_1 e^{-0} + c_2 \left(e^{-0} + 0 \cdot (-e^{-0}) \right)$$

$$= -c_1 + c_2 \quad , \text{ so } \quad c_2 = 1 + c_1 = 2,$$

and $y = e^{-x} + 2xe^{-x}$.

To find a maximum, we look for critical points:

$$y' = -e^{-x} + (2e^{-x} - 2xe^{-x}) = e^{-x}(1-2x)$$

is always defined, and is 0 only when $x = \frac{1}{2}$.

At $x=0$, $y=1$, and as $x \to \infty$, $y \to 0$, so the critical point at $x=\frac{1}{2}$,

with $y = e^{-1/2} + 2 \cdot \frac{1}{2} \cdot e^{-1/2} \approx 1.213$, is the absolute maximum.

See accompanying figure for the plot.