MAT 2384
HW #1 solutions

1.8 This is a separable D.E.; to solve it we separate the x's and y's. First write this as

$$x \sin y \, dx = -(x^2+1) \cos y \, dy \, ;$$

then algebraically reduce to

$$\frac{\cos y}{\sin y} \, dy = -\frac{x}{x^2+1} \, dx \qquad (*)$$

(Now separated.)
Add in an integral sign

$$\int \frac{\cos y}{\sin y} \, dy = \int -\frac{x}{x^2+1} \, dx$$

and evaluate

$$\ln |\sin y| = -\frac{1}{2} \ln (x^2+1) + C \qquad C \text{ any constant}$$

(note that x^2+1 is always positive, so $|x^2+1| = x^2+1$.)

Take exponentials to get

$$|\sin y| = e^{C} (x^2+1)^{-1/2}, \qquad \sin y = \pm e^{C} (x^2+1)^{-1/2}$$

Comment:

~~was~~ (To be extra careful we should notice that back in (*) we divided by $\sin y$, and this isn't permitted if $\sin y = 0$. You can verify that setting y to be any constant with $\sin y = 0$ (e.g. $y(x) = 0$, or $y(x) = \pi$) also gives a solution to the D.E.).

Let us write $k = \pm e^C$ (or $= 0$, as in the comment above.) Then k can be an arbitrary constant. (This is a common trick, and as a bonus gives the extra solutions mentioned in the comment)

So the general solution is given by

$$\sin y = k(x^2+1)^{-1/2} \quad , \quad k \text{ any constant.}$$

Comment: since $\sin$ is periodic, there isn't a unique solution for y, given x. The general form would be

$$y = \begin{cases} \arcsin\left(k(x^2+1)^{-1/2}\right) + 2\pi n, & \text{or} \\ \pi - \arcsin\left(k(x^2+1)^{-1/2}\right) + 2\pi n, & \end{cases}$$

k any (real number) constant
n any integer ($n = 0, \pm 1, \pm 2$, etc.)

We're given an IC, let's use it to solve for y.

Start with $\sin y = k(x^2+1)^{-1/2}$.

When $x=1$, $y=\frac{\pi}{2}$, so

$$\underbrace{\sin(\tfrac{\pi}{2})}_{=1} = \underbrace{k(1^2+1)^{-1/2}}_{=k/\sqrt{2}}$$

so $k=\sqrt{2}$,

$$\sin y = \sqrt{2}(x^2+1)^{-1/2} = \sqrt{\frac{2}{x^2+1}}.$$

Solving for y gives

$$\boxed{y = \arcsin\left(\sqrt{\frac{2}{x^2+1}}\right)}$$

as the solution defined for $x \geq 1$. (Note that it is not defined for any positive x less than 1.)

The attached graph is directly from google "plot $y=\arcsin(\cdots)$".

1.12 This equation is homogeneous of degree 1: multiplying by dx gives (recall $y' = \frac{dy}{dx}$)

$$(y + x\cos^2(y/x))\,dx - x\,dy = 0.$$

Both $M(x,y) = y + x\cos^2(y/x)$ and $N(x,y) = -x$ are homogeneous of degree 1, since $M(\lambda x, \lambda y) = \lambda y + \lambda x \cos^2\left(\frac{\lambda y}{\lambda x}\right)$

$$= \lambda y + \lambda x \cos^2\left(\frac{y}{x}\right)$$

$$= \lambda\left(y + x\cos^2\left(\frac{y}{x}\right)\right)$$

$$= \lambda^1 M(x,y) ,$$

and similarly with $N(x,y)$. Based on the form of the DE, we guess that $v = y/x$ will lead to an easier equation than $v = x/y$, so we set $v = y/x$.

$y = vx$, $dy = v\,dx + x\,dv$, so replacing y with vx and dy with $v\,dx + x\,dv$ changes the DE to

$$(vx + x\cos^2(v))\,dx + (-x)(v\,dx + x\,dv) = 0 .$$

Collecting dx and dv:

$$(x\cos^2 v)\,dx - (x^2)\,dv = 0 , \qquad x\cos^2 v\,dx = x^2\,dv .$$

Separating,

$$\int \frac{dv}{\cos^2 v} = \int \frac{dx}{x} .$$

$$\int \sec^2 v\,dv = \ln|x| + C$$

$$\tan v$$

Expressing v back in terms of x and y:

$$\tan(y/x) = \ln|x| + C$$

Comment: Solving for y gives $y = x(\arctan(\ln|x| + C) + n\pi)$,

C any number
n any integer.

1.29 Rewrite $(1-xy)y' + y^2 + 3xy^3 = 0$ as

$$(y^2 + 3xy^3)dx + (1-xy)dy = 0.$$

This is not separable, not homogeneous coefficients. Exact?

$$M_y = \frac{\partial}{\partial y}(y^2 + 3xy^3) = 2y + 9xy^2$$

$$N_x = \frac{\partial}{\partial x}(1-xy) = -y. \neq M_y \qquad \text{Not exact.}$$

$$M_y - N_x = 3y + 9xy^2 = 3(y + 3xy^2) = 3y(1+3xy)$$

So $\dfrac{M_y - N_x}{M} = \dfrac{3y(1+3xy)}{y^2(1+3xy)} = \dfrac{3}{y}$ depends only on y, call it $g(y)$.

Then $u(y) = e^{-\int g(y)dy} = e^{-\int \frac{3}{y}dy} = e^{-3\ln y} = y^{-3}$ is an integrating factor.

Multiply both sides by $u(y)$ to get

$$(y^{-1} + 3x)dx + (y^{-3} - xy^{-2})dy = 0$$

This new D.E. can be verified to be exact, and it has the same solution as the original DE. (except for the function $y \equiv 0$.) We find ϕ in the usual two step process:

1) $\phi_x = \tilde{M}(x,y) = y^{-1} + 3x$, so

$\phi(x,y) = xy^{-1} + \frac{3}{2}x^2 + h(y)$, for some (to be determined) function $h(y)$ that depends only on y.

2) Using this, $\phi_y = -xy^{-2} + 0 + h'(y)$; on the other hand,

$$\phi_y = N(x,y) = y^{-3} - xy^{-2}, \quad \text{so}$$

$$-xy^{-2} + h'(y) = y^{-3} - xy^{-2}, \quad h'(y) = y^{-3}$$

A solution to this is $h(y) = -\frac{y^{-2}}{2}$, this gives

$$\phi(x,y) = xy^{-1} + \frac{3}{2}x^2 - \frac{y^{-2}}{2}.$$

The general solution is then

$$xy^{-1} + \frac{3}{2}x^2 - \frac{y^{-2}}{2} = C, \qquad C \text{ any constant.}$$

$$y^{-2} - 2xy^{-1} + (-3x^2 + 2C) = 0$$

$$y^{-1} = \frac{2x \pm \sqrt{4x^2 - 4(-3x^2+2C)}}{2} = x \pm \sqrt{4x^2 - 2C}$$

$$y = \frac{1}{x \pm \sqrt{4x^2 - 2C}} \qquad C \text{ any constant}$$

(together with $y \equiv 0$)

1.38 If we divide through by x, then we have a linear D.E. in standard form:

$$y' - \frac{2}{x}y = 2x^3 \quad , \qquad \text{i.e. } y' + f(x)y = r(x), \text{ with } f(x) = -\frac{2}{x}, \ r(x) = 2x^3.$$

(Alternatively, you can find an integrating factor for the original D.E.)

The general solution is then

$$y(x) = \frac{\int e^{\int f(x)dx} r(x)dx + C}{e^{\int f(x)dx}} .$$

Here $e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$, so

$$y(x) = \frac{\int (x^{-2})(2x^3)dx + C}{x^{-2}}$$

$$= x^2\left(\int 2x\,dx + C\right) = x^2(x^2 + C), \quad C \text{ any constant.}$$

We're given the IC $y(2) = 8$, so

$$8 = 2^2(2^2 + C) = 4(4 + C),$$

$$2 = 4 + C, \qquad C = -2.$$

So $y = x^2(x^2 - 2) = x^4 - 2x^2$.

7.1 We begin with $a_0 = 0$, $b_0 = 1$. The first midpoint, x_1, is $\frac{a_0+b_0}{2} = 0.5$.

$f(a_0) = f(0) = \sqrt{0} - \cos(0) = -1 \quad < 0$

$f(b_0) = f(1) = \sqrt{1} - \cos(1) \sim 0.45970$ (rounded to 5 significant digits) > 0.

$f(x_1) = f(0.5) = \sqrt{0.5} - \cos(0.5) \sim -0.17048 \; < 0$,

so we set $a_1 = x_1 = 0.5$, $b_1 = b_0 = 1$. Note that $f(a_1) < 0$, $f(b_1) > 0$, as with a_0 and b_0.

The next midpoint is $x_2 = \frac{a_1 + b_1}{2} = \frac{0.5+1}{2} = 0.75$

$f(x_2) = f(0.75) = \sqrt{0.75} - \cos(0.75) \sim 0.13434 \; > 0$, so we set

$a_2 = a_1 = 0.5$, $b_2 = x_2 = 0.75$. Again, $f(a_2) < 0$, $f(b_2) > 0$.

The last midpoint is $x_3 = \frac{a_2 + b_2}{2} = \frac{0.5 + 0.75}{2} = 0.625$

7.8. All we do here is evaluate the function g several times.

$x_1 = g(\cancel{x_0}) = g(2) = \cos(2-1) = \cos(1) \sim 0.5403023059$

$x_2 = g(x_1) \sim \cos(0.5403023058 - 1) \sim 0.8961866647$

$x_3 = g(x_2) \sim 0.9946162335$

$x_4 = g(x_3) \sim 0.9999855076$

$x_5 = g(x_4) \sim 0.999999999$.

In this case, we know that the sequence approaches the fixed point $p = 1$ (note $g(1) = \cos(1-1) = \cos(0) = 1 = p$)

The errors are given by

$\varepsilon_1 = x_1 - 1 \sim -0.4596976941$

$\varepsilon_2 = x_2 - 1 \sim -0.1038133353$

$\varepsilon_3 = x_3 - 1 \sim -0.0053837665$

$\varepsilon_4 = x_4 - 1 \sim -0.0000144924$

$\varepsilon_5 = x_5 - 1 \sim -1 \times 10^{-10}$ (one significant digit, 10 decimal place accuracy)

We will see that the sequence $\frac{\varepsilon_2}{\varepsilon_1}, \frac{\varepsilon_3}{\varepsilon_2}, \frac{\varepsilon_4}{\varepsilon_3}, \frac{\varepsilon_5}{\varepsilon_4}, \ldots$

will converge to 0, but the sequence

$\frac{\varepsilon_2}{(\varepsilon_1)^2}, \frac{\varepsilon_3}{(\varepsilon_2)^2}, \frac{\varepsilon_4}{(\varepsilon_3)^2}, \frac{\varepsilon_5}{(\varepsilon_4)^2}, \ldots$, numerically

approaches 0.5 (a nonzero limit.)

So we predict an order of convergence of 2.

(the exponent in the denominator)

We can verify this by looking at the derivatives of g.

$g'(x) = -\sin(x-1)$, so $g'(p) = g'(1) = -\sin(0) = 0$.

$g''(x) = -\cos(x-1)$, so $g''(p) = g''(1) = -\cos(0) = -1$.

The first nonzero derivative at $p=1$ is the second, so the convergence of second order, as predicted.

7.10 This is another fixed point method, with

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{2x - \tan x}{2 - \sec^2 x}.$$

$x_1 = g(x_0) = g(1) \sim 1.3104780301$

$x_2 = g(x_1) \sim 1.2239290965$
$x_3 = g(x_2) \sim 1.1760509000$
$x_4 = g(x_3) \sim 1.1659265083$
$x_5 = g(x_4) \sim 1.1655616365$
$x_6 = g(x_5) \sim 1.1655611852$
$x_7 = g(x_6) \sim 1.1655611852$

Once the new entry has the same first 6 decimal values as the previous, we are done.

So, to 6 decimals the root is 1.165561.

For the order of convergence we can look at the errors, or use the theoretical method. We'll use the theoretical method.

Notice that on page 170 of the notes, it's shown that if p is a fixed point and f''' exists, then

$$g''(p) = -\frac{f''(p)}{f'(p)}$$

The formula for $f'(x)$ is $2 - \sec^2 x = 2 - (\cos x)^{-2}$
and the formula for $f''(x)$ is then $-2(\cos x)^{-3}(-\sin x)$
$= 2 \tan x \sec^2 x$.

We don't know the exact value of p, but it's close to 1.165561

So $g''(p) = -\frac{f''(p)}{f'(p)} \sim -\frac{f''(1.165561)}{f'(1.165561)}$

~ 6.765133

We can be pretty sure that $g''(p)$ is non-zero. (6.7 is not within rounding errors of 0)

By Theorem 7.7, we know that $g'(p) = 0$ and Newton's method has order at least 2.

Since $g''(p) \neq 0$, we know its order is $\underline{2}$.