

DEPARTMENT OF MATHEMATICS
MID TERM TEST 2 - SOLUTIONS
MTH 140 – CALCULUS I

Date: October 31st, 2014, 4:00 pm

Duration: 1.5 hours

Last Name: _____

First Name: _____

Ryerson email: _____

Student Number(ONLY LAST 3 DIGITS): _____

Signature: _____

Section (circle one)

A. Alvarez :	1	2	3	4	
D. Ha :	6	7	8	9	10
X. Liu :	11	12	13	14	15
N. Jung :	16	17	18	19	20
J. Horowitz :	21	22	23	24	25

Instructions:

1. This is a closed-book test. **Notes, calculators and other aids are not permitted.**
 2. Verify that your test has pages 1-10. The exam questions are in pages 2-8. Pages 9 and 10 are given for extra space, they do NOT contain any questions.
 3. (a) Unless otherwise instructed, **make sure you include all significant steps in your solution, presented in the correct order. Unjustified answers will be given little or no credit. Cross out or erase all rough work not relevant to your solution.** Put a box around your final **answer.**
 - (b) For multiple choice questions make sure to write your answers in the box at the end of each question **carefully.** There are no part marks in the multiple-choice section and **only** the answer in the box will be marked. The correct response gets full marks, an incorrect response or no response gets no marks.
 - (c) Write your solutions in the space provided. **ANYTHING WRITTEN IN THE BACK OF ANY PAGE WILL NOT BE MARKED.** If you need more space, use the front part of page 9 and page 10. Indicate this fact on the original page.
4. Do not separate the sheets.
 5. Have your student card available on your desk.

1. [2 marks] (Multiple choice question). The limit $\lim_{x \rightarrow 4} \frac{\sin(x-4)}{\sqrt{x}-2}$ is equal to:

- A) 2 B) 1 C) 4 D) The limit does not exist
E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked.

2. [2 marks] (Multiple choice question). The derivative of $f(x) = (\tan x)^{1/x}$ is:

- A) $f'(x) = \frac{(\tan x)^{1/x-1}}{x}$ B) $f'(x) = (\tan x)^{1/x} \left[\frac{\sec^2 x}{x \tan x} - \frac{\ln \tan x}{x^2} \right]$
C) $f'(x) = (\tan x)^{1/x} \left[\frac{\sec^2 x}{x \tan x} + \frac{\ln \tan x}{x^2} \right]$ D) $f'(x) = (\tan x)^{1/x} \cdot \ln \tan x$
E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked.

3. [2 marks] (Multiple choice question). Let $f(x) = \sqrt[3]{x}$. Using the linearization $L(x)$ of f at $a = 1$, an approximation of $\sqrt[3]{1.03}$ is:

- A) 0.03 B) 1.03 C) $\frac{1}{1.03}$ D) 2.03 E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked.

4. [2 marks] . The second derivative of $f(x) = \sin(x^2 + 1)$ is:

- A) $f''(x) = -\sin(x^2 + 1)$ B) $f''(x) = -4x^2 \sin(x^2 + 1)$
C) $f''(x) = 2 \cos(x^2 + 1) + 4 \sin(x^2 + 1)$
D) $f''(x) = 2 \cos(x^2 + 1) - 4x^2 \sin(x^2 + 1)$ E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked.

5. [6 marks] Given the function $f(x) = \frac{x}{2x-1}$

- a) Find the derivative of f using the definition of the derivative as a limit.
b) Find the derivative of f using the rules of differentiation.

Solution

$$\begin{aligned} \text{a) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{x+h}{2(x+h)-1} - \frac{x}{2x-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{(x+h)(2x-1) - x(2(x+h)-1)}{(2(x+h)-1)(2x-1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{(2x^2+2xh-x-h)-(2x^2+2xh-x)}{(2(x+h)-1)(2x-1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{(2(x+h)-1)(2x-1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h(2(x+h)-1)(2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(2(x+h)-1)(2x-1)} \\ &= \frac{-1}{(2(x+0)-1)(2x-1)} \\ &= \frac{-1}{(2x-1)^2} \\ \text{b) } f'(x) &= \frac{(x)'(2x-1) - x(2x-1)'}{(2x-1)^2} = \frac{1 \cdot (2x-1) - x \cdot 2}{(2x-1)^2} = -\frac{1}{(2x-1)^2} \end{aligned}$$

6. [7 marks] Find the limit

$$\lim_{x \rightarrow -\infty} \left(2x + \sqrt{4x^2 - 8x} \right)$$

Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left(2x + \sqrt{4x^2 - 8x} \right) &= \lim_{x \rightarrow -\infty} \left(2x + \sqrt{4x^2 - 8x} \right) \cdot \frac{\left(2x - \sqrt{4x^2 - 8x} \right)}{\left(2x - \sqrt{4x^2 - 8x} \right)} \\ &= \lim_{x \rightarrow -\infty} \frac{4x^2 - (4x^2 - 8x)}{\left(2x - \sqrt{4x^2 - 8x} \right)} = \lim_{x \rightarrow -\infty} \frac{8x}{\left(2x - \sqrt{4x^2 - 8x} \right)} \end{aligned}$$

Dividing by x both the numerator and denominator we get

$$= \lim_{x \rightarrow -\infty} \frac{8}{\left(2 - \frac{\sqrt{4x^2 - 8x}}{x} \right)}$$

As x is negative, $x = -\sqrt{x^2}$ therefore

$$= \lim_{x \rightarrow -\infty} \frac{8}{\left(2 + \sqrt{\frac{4x^2 - 8x}{x^2}} \right)} = \lim_{x \rightarrow -\infty} \frac{8}{\left(2 + \sqrt{4 - \frac{8}{x}} \right)} = \frac{8}{2 + 2} = 2$$

7. [8 marks] Find equations of the tangent line and normal line to the curve $y = 2^{x^2-x-1} \sec x$ at the point $\left(0, \frac{1}{2}\right)$

Solution:

$$y' = 2^{x^2-x-1} \sec x \tan x + \sec x \cdot 2^{x^2-x-1} \cdot \ln 2 \cdot (2x-1)$$

$$\text{Therefore } y'(0) = 2^{-1} \sec 0 \tan 0 + \sec 0 \cdot 2^{-1} \cdot \ln 2 \cdot (-1)$$

Using that $\sec 0 = 1$ and $\tan 0 = 0$ we get

$$y'(0) = -\frac{\ln 2}{2} = m_1$$

$$\text{Equation of tangent line: } y - \frac{1}{2} = -\frac{\ln 2}{2}x$$

$$\text{The slope of the normal line is } m_2 = -\frac{1}{m_1} = \frac{2}{\ln 2}$$

$$\text{Equation of normal line: } y - \frac{1}{2} = \frac{2}{\ln 2}x$$

8. [6 marks] Let f be defined as

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x - 3)^2 & \text{if } x \geq 3 \end{cases}$$

a) Is the function f continuous at $a = 0$? Explain.

b) Is the function f continuous at $a = 3$? Explain.

Solution:

a)

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sqrt{-x} = \sqrt{0} = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (3 - x) = 3 - 0 = 3$$

As $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ we can conclude that $\lim_{x \rightarrow 0} f(x)$ DOES NOT EXIST therefore the function f is NOT CONTINUOUS at $a = 0$.

b)

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (3 - x) = 3 - 3 = 0$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x - 3)^2 = (3 - 3)^2 = 0$$

Additionally, we have that $f(3) = (3 - 3)^2 = 0$

As both lateral limits are equal we conclude that $\lim_{x \rightarrow 3} f(x)$ exists and is equal to 0. Also as

$\lim_{x \rightarrow 3} f(x) = f(3) = 0$ we conclude that the function IS CONTINUOUS at $a = 3$

9. [7 marks] Find $\frac{dy}{dx}$ by implicit differentiation

$$e^{5y} \sinh x = x^3 + \tanh y$$

Solution:

$$\frac{d}{dx}(e^{5y} \sinh x) = \frac{d}{dx}(x^3 + \tanh y)$$

$$e^{5y} \cosh x + \sinh x \cdot e^{5y} \cdot 5 \frac{dy}{dx} = 3x^2 + \operatorname{sech}^2 y \frac{dy}{dx}$$

$$5 \sinh x \cdot e^{5y} \frac{dy}{dx} - \operatorname{sech}^2 y \frac{dy}{dx} = 3x^2 - e^{5y} \cosh x$$

$$\frac{dy}{dx} (5 \sinh x \cdot e^{5y} - \operatorname{sech}^2 y) = 3x^2 - e^{5y} \cosh x$$

Then,

$$\frac{dy}{dx} = \frac{3x^2 - e^{5y} \cosh x}{(5 \sinh x \cdot e^{5y} - \operatorname{sech}^2 y)}$$

10. [8 marks] A particle moves along the curve $y = 2 \sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s.

How fast is the distance from the particle to the origin changing at this instant?

Solution:

$$s^2 = x^2 + y^2, \text{ then } 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

and

$$\frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt} + \frac{y}{s} \frac{dy}{dt}$$

At the point $\left(\frac{1}{3}, 1\right)$ we have that

$$s = \sqrt{1 + \frac{1}{9}} = \frac{\sqrt{10}}{3}$$

Also

$$\frac{dy}{dt} = 2 \cos(\pi x/2) \cdot \frac{\pi}{2} \cdot \frac{dx}{dt} = \pi \cos(\pi x/2) \cdot \frac{dx}{dt}$$

$$\left. \frac{dy}{dt} \right|_{x=1/3} = \pi \cos(\pi/6) \cdot \sqrt{10} = \frac{\sqrt{3}\pi}{2} \cdot \sqrt{10}$$

Then

$$\begin{aligned} \left. \frac{ds}{dt} \right|_{x=1/3} &= \frac{1/3}{\sqrt{10}/3} \sqrt{10} + \frac{1}{\sqrt{10}/3} \frac{\sqrt{3}\pi}{2} \cdot \sqrt{10} \\ &= \left(1 + \frac{3\sqrt{3}\pi}{2}\right) \text{ cm/s} \end{aligned}$$