ENGR 213/2, Section XX, Fall 2013, Midterm Test 2, Solutions

Problem 1. The differential equation

$$(1+x^2)y'' - 2xy' + 2y = 0$$

has a solution $y_1(x) = x$. Find another solution $y_2(x)$ such that the solution set $\{y_1(x), y_2(x)\}$ is a fundamental set of solutions of the given differential equation for x in $(-\infty, \infty)$.

Problem 2. Find the general solution of the differential equation

$$y'' + 2y' + y = e^{-x}$$

(a) by using the method of undetermined coefficients;
(b) by using the method of variation of parameters.

Problem 3. Solve the differential equation

$$x^2y'' + xy' - y = \ln(x), \quad x > 0.$$

Problem 4. A mass of 1 kg is attached to a spring whose spring constant is 1 N/m. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially **released from rest**, 20 cm below the equilibrium position.
(a) Formulate an initial value problem (that is: a differential equation and initial value conditions) describing the dynamics of the spring-mass system;
(b) Characterize the spring-mass system as under-, over-, or critically damped;
(c) Find an equation of the spring-mass motion.
(d) Represent the equation obtained in **(c)** in **an amplitude-phase form**;
(e) Find the first time-moment at which the mass passes through the equilibrium position. What is the velocity of the mass at this time-moment - upward or downward?

Problem 5. (a) Find the first five coefficients (i.e. a_0, a_1, a_2, a_3, a_4) of the power series expansion

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

centered at $x_0 = 0$ of the solution $y(x)$ of the following second-order initial value problem

$$(x-2)y'' + y = 0, \quad y(0) = 1, \ y'(0) = 0.$$

(b) Find all singular points of the initial value problem given in **(a)** and determine the minimum radius of convergence of the power series solution $y(x)$ obtained in **(a)**.

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Solutions

① Solution 1. First, standard form

$$y'' - \frac{2x}{1+x^2}\, y' + \frac{2}{1+x^2}\, y = 0$$

$$p(x) = -\frac{2x}{1+x^2}, \quad y_1(x) = x.$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)}\, dx = x \int \frac{e^{\int \frac{2x}{1+x^2} dx}}{x^2}\, dx$$

$$= x \int \frac{e^{\ln(1+x^2)}}{x^2}\, dx = x \int \frac{1+x^2}{x^2}\, dx$$

$$= x \int (x^{-2}+1)\, dx = x\left(-x^{-1}+x\right) = x^2 - 1$$

Check: $y_1(x) = x$

$$(1+x^2)\, y_1'' - 2x\, y_1' + 2\, y_1 = -\cancel{2x} + \cancel{2x} = 0 \quad (\text{o.k.})$$

$$y_2(x) = x^2 - 1$$

$$(1+x^2)\, y_2'' - 2x\, y_2' + 2\, y_2 = (1+x^2)(2) - 2x(2x) + 2(x^2-1)$$

$$= \cancel{2} + \cancel{2x^2} - \cancel{4x^2} + \cancel{2x^2} - \cancel{2} = 0 \quad (\text{o.k.}$$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix} = 2x^2-(x^2-1)$$

$$\boxed{\begin{array}{l} y_1 = x \\ y_2 = x^2-1 \end{array}} \qquad = x^2+1 \neq 0$$

for all $x \in (-\infty, \infty) \Rightarrow$

$\{y_1, y_2\}$ is a fundamental set of solutions.

Anothermethod: $c_1 x + c_2(x^2-1) = 0$, $x \in (-\infty, \infty)$

$\Rightarrow c_2 x^2 + c_1 x - c_2 = 0 \Rightarrow c_1 = 0, c_2 = 0$

$\Rightarrow \{y_1, y_2\}$ are linearly independent $\Rightarrow \{y_1, y_2\}$ is a fundamental set in $(-\infty, \infty)$.

Solution 2. $y_2(x) = y_1(x)u(x) = xu$

$(1+x^2)\left((x)''u + 2(x)'(u)' + xu''\right) - 2x(u + xu')$

$+ 2xu = 0$

$(1+x^2)(2u' + xu'') - \cancel{2xu} - 2x^2u' + \cancel{2xu} = 0$

$\Rightarrow (1+x^2)xu'' + (2 + \cancel{2x^2} - \cancel{2x^2})u' = 0$

$\Rightarrow (1+x^2)xu'' + 2u' = 0$; $\boxed{v = u'}$

$\Rightarrow (1+x^2)xv' + 2v = 0$

$$(1+x^2)x\,v' = -2v \Rightarrow \frac{dv}{v} = -\frac{2}{(1+x^2)x}\,dx$$

$$\Rightarrow \int \frac{dv}{v} = -\int \frac{2x}{(1+x^2)x^2}\,dx \Rightarrow \int \frac{dv}{v} = -\int \frac{1}{(1+x^2)x^2}\,dx^2$$

$x^2 = w$

$$\int \frac{1}{(1+x^2)x^2}\,dx^2 = \int \frac{dw}{(1+w)w} = \int \left(\frac{1}{w} - \frac{1}{1+w}\right) dw$$

$$= \ln \frac{w}{1+w} = \ln \frac{x^2}{1+x^2} \Rightarrow \left(-\ln \frac{x^2}{1+x^2} = \ln \frac{1+x^2}{x^2}\right)$$

$$\Rightarrow \ln|v| = \ln \frac{1+x^2}{x^2} \Rightarrow v = x^{-2} + 1$$

$$\Rightarrow u' = x^{-2} + 1 \Rightarrow u = \int (x^{-2}+1)\,dx = -x^{-1} + x$$

$$\Rightarrow y_2(x) = x\,u(x) = x(-x^{-1}+x) = x^2 - 1.$$

Next, by using the same methods as in the Solution 1, we get that $\{y_1, y_2\}$ is a fundamental set of solutions.

② (a) by undetermined coefficients

(A) Solve the associated homogeneous d.e.

$$y''+2y'+y=0$$

$m^2+2m+1=0$, $m_1=m_2=-1 \Rightarrow y_1=e^{-x}$, $y_2=xe^{-x}$

$$y_c = C_1e^{-x}+C_2xe^{-x}.$$

(B) $f(x)=e^{-x}$, $f'(x)=-e^{-x}$, $f''(x)=e^{-x}$, ...

$\Rightarrow$ $y_p=Ae^{-x}$ (avoid the effect of duplication)

$\Rightarrow$ $\boxed{y_p=Ax^2e^{-x}}$

$y_p=Ax^2e^{-x}$; $y_p'=A(2x-x^2)e^{-x}$;

$y_p''=A(2-2x-2x+x^2)e^{-x}=(2-4x+x^2)e^{-x}$.

$\Rightarrow y_p''+2y_p'+y_p=A(2-4x+x^2+4x-2x^2+x^2)e^{-x}$

$=2Ae^{-x}=e^{-x} \Rightarrow 2A=1 \Rightarrow A=\frac{1}{2} \Rightarrow$

$\boxed{y_p=\frac{x^2}{2}e^{-x}}$ $\quad y=y_c+y_p$ (the general solution)

The general solution:

$$y=\underbrace{C_1e^{-x}+C_2xe^{-x}}_{y_c}+\underbrace{\frac{x^2}{2}e^{-x}}_{y_p}$$

(b) by variation of parameters

From 2 (a) : $y_c = C_1 y_1 + C_2 y_2$; $y_1 = e^{-x}$, $y_2 = xe^{-x}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$$

$$= (1-x)e^{-2x} + xe^{-2x} = e^{-2x}.$$

Standard form : $f(x) = e^{-x}$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x),$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & xe^{-x} \\ e^{-x} & (1-x)e^{-x} \end{vmatrix}}{e^{-2x}} = -\frac{xe^{-2x}}{e^{-2x}} = -x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \end{vmatrix}}{e^{-2x}} = \frac{e^{-2x}}{e^{-2x}} = 1$$

$u_1'(x) = -x$; $u_1(x) = \int -x \, dx = -\frac{x^2}{2}$,

$u_2'(x) = 1$, $u_2(x) = \int 1 \, dx = x$

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2 = -\frac{x^2}{2} e^{-x} + x \cdot (xe^{-x})$$

$$= \frac{x^2}{2} e^{-x} \qquad \boxed{y_p(x) = \frac{x^2}{2} e^{-x}}$$

$$y = y_c + y_p \Rightarrow y = \underbrace{C_1 e^{-x} + C_2 x e^{-x}}_{y_c} + \underbrace{\frac{x^2}{2} e^{-x}}_{y_p}.$$

3 Cauchy - Euler d.e.

(A) Solve the associated homogeneous d.e.

$$x^2 y'' + x y' - y = 0, \quad x > 0 \; ; \quad y = x^m$$

$$x^2 m(m-1) x^{m-2} + \cancel{x} m x^{m-1} - x^m = 0, \; x > 0$$

$$\Rightarrow m(m-1) x^m + m x^m - x^m = 0$$

$$\Rightarrow \cancel{x^m} \left[m(m-1) + m - 1 \right] = 0 \Rightarrow m(m-1) + m - 1 = 0$$

$$\Rightarrow m^2 - m + m - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m_{1,2} = \pm 1.$$

$$m_1 = 1, \; m_2 = -1 \; ; \quad y_1 = x, \; y_2 = x^{-1} = \frac{1}{x},$$

$$y_c = c_1 y_1 + c_2 y_2 \Rightarrow \boxed{y_c = c_1 x + c_2 x^{-1}}$$

(B) Find y_p by variation of parameters:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}.$$

Standard form to determine $f(x)$:

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = \frac{\ln(x)}{x^2} \Rightarrow f(x) = \frac{\ln(x)}{x^2}.$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^{-1} \\ \frac{\ln(x)}{x^2} & -x^{-2} \end{vmatrix}}{-2x^{-1}} = -\frac{\ln(x)\, x^{-3}}{-2x^{-1}} = \frac{x^{-2}}{2} \ln(x).$$

$$u_1(x) = \int \frac{x^{-2} \ln(x)}{2} dx = -\frac{x^{-1} \ln(x)}{2} + \int \frac{x^{-1}}{2} \cdot \frac{1}{x} dx$$

(by parts)

$$= -\frac{x^{-1}}{2} \ln(x) + \frac{1}{2} \int x^{-2} dx$$

$$u_1(x) = -\frac{x^{-1}}{2}\ln(x) + \frac{1}{2}\,\frac{x^{-1}}{-1} = -\frac{x^{-1}}{2}\ln(x) - \frac{1}{2}x^{-1},$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{\ln(x)}{x^2} \end{vmatrix}}{-2x^{-1}} = \frac{x^{-1}\ln(x)}{-2x^{-1}} = -\frac{\ln(x)}{2}.$$

$$u_2(x) = \int -\frac{\ln(x)}{2}\,dx \underset{\text{(by parts)}}{=} -\frac{x\ln(x)}{2} + \int x\cdot\frac{1}{2x}\,dx$$

$$u_2(x) = -\frac{x\ln(x)}{2} + \frac{1}{2}x.$$

$$y_p(x) = y_1(x)u_1(x) + y_2(x)u_2(x) = x\left(-\frac{x^{-1}}{2}\ln(x) - \frac{x^{-1}}{2}\right)$$
$$+\ x^{-1}\left(-\frac{x\ln(x)}{2} + \frac{x}{2}\right) = -\frac{\ln(x)}{2} - \cancel{\frac{1}{2}} - \frac{\ln(x)}{2} + \cancel{\frac{1}{2}} = -\ln(x)$$

$$\Rightarrow \boxed{y_p(x) = -\ln(x)}$$

The general solution:

$$y = y_c + y_p$$

$$\boxed{y = \underbrace{C_1 x + C_2 x^{-1}}_{y_c(x)} \underbrace{- \ln(x)}_{y_p(x)}}$$

(4) $x = x(t)$, $m x'' + \beta x' + K x = 0$

$m = 1$, $\beta = 1$, $K = 1$.

(a) $\left| \begin{array}{l} x'' + x' + x = 0 \\ x(0) = 0.2\ m;\ x'(0) = 0\ m/s \end{array} \right.$

(released from rest)

Second-order IVP.

(b) $m^2 + m + 1 = 0$; $D = 1^2 - 4(1)(1) = -3 < 0$

$\Rightarrow$ under-damped spring-mass system.

(c) $m^2 + m + 1 = 0$; $m_{1,2} = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$$m_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x(t) = e^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$x(0) = 0.2$; $x'(0) = 0$.

$x(0) = C_1 = 0.2 = \frac{1}{5}$.

$$x'(t) = -\frac{1}{2}e^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$
$$+ e^{-t/2}\left(-C_1\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) + C_2\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$x'(0) = -\frac{1}{2}C_1 + C_2\frac{\sqrt{3}}{2} = 0 \Rightarrow C_2 = \frac{C_1}{\sqrt{3}} = \frac{\sqrt{3}}{15}.$$

$$\Rightarrow \boxed{x(t) = e^{-t/2}\left(\frac{1}{5}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{15}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}$$

or equivalently:

$$\boxed{x(t) = 0.2e^{-t/2}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right).}$$

(d) amplitude-phase form:

$$C_1 = 1, \quad C_2 = \frac{1}{\sqrt{3}} \quad ; \quad C_1^2 + C_2^2 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\sqrt{C_1^2 + C_2^2} = \frac{2}{\sqrt{3}} \quad \Rightarrow$$

$$x(t) = \frac{0.4}{\sqrt{3}} e^{-t/2}\left(\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \Rightarrow \boxed{\phi = \frac{\pi}{3}} \quad \boxed{\phi = 60^\circ}$$

$$\Rightarrow x(t) = \frac{0.4}{\sqrt{3}} e^{-t/2}\left(\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\sqrt{3}}{2}t\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$\Rightarrow \boxed{x(t) = \frac{0.4}{\sqrt{3}} e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)}$$

or equivalently

$$x(t) = \frac{(0.4)(\sqrt{3})}{3} e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)$$

(e) $\frac{\sqrt{3}}{2}t + \frac{\pi}{3} = \pi \Rightarrow \frac{\sqrt{3}}{2}t = \frac{2\pi}{3}$; $\boxed{t_0 = \frac{4\pi}{3\sqrt{3}}}$

or equivalently: $\boxed{t_0 = \frac{4\sqrt{3}\pi}{9}}$.

The velocity at t_0 will be upward, i.e., negative.

$$x'(t) = \frac{0.4}{\sqrt{3}}\left(-\frac{1}{2}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right) + e^{-t/2}\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)\right).$$

$$x'(t_0) = x'\left(\frac{4\pi}{3\sqrt{3}}\right) = \frac{0.4}{\sqrt{3}}e^{-t_0/2} \times \left(-\frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}\cdot\frac{4\pi}{3\sqrt{3}} + \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}\frac{4\pi}{3\sqrt{3}} + \frac{\pi}{3}\right)\right)$$

$$= \frac{0.4}{\sqrt{3}}e^{-t_0/2}\left(-\frac{1}{2}\sin\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)\right)$$

$$= \frac{0.4}{\sqrt{3}}e^{-\frac{2\pi}{3\sqrt{3}}}\left(-\frac{1}{2}\overset{(0)}{\sin(\pi)} + \frac{\sqrt{3}}{2}\overset{(-1)}{\cos(\pi)}\right)$$

$$= -\frac{0.4}{\sqrt{3}}e^{-2\pi/3\sqrt{3}}$$

$$x'(t_0) = -\frac{0.4}{\sqrt{3}}e^{-\frac{2\pi}{3\sqrt{3}}} \text{ m/s}$$

$\boxed{\text{upward negative velocity}}$

(5) (a) Solution 1. By changing of the summation index:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n; \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(x-2) y'' + y = (x-2)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - 2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n - 2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(a_0 - 4a_2) + \sum_{n=1}^{\infty} \left((n+1) n a_{n+1} - 2(n+2)(n+1) a_{n+2} + a_n\right) x^n = 0$$

$$\Rightarrow a_2 = \frac{a_0}{4}; \quad a_{n+2} = \frac{(n+1) n a_{n+1} + a_n}{2(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow a_2 = \frac{a_0}{4}, \quad a_{n+2} = \frac{n}{2(n+2)} a_{n+1} + \frac{1}{2(n+2)(n+1)} a_n,$$

$$n = 1, 2, 3, \ldots$$

$y(0) = 1 \Rightarrow \boxed{a_0 = 1}$; $y'(0) = 0 \Rightarrow \boxed{a_1 = 0}$

$$a_2 = \frac{a_0}{4} = \frac{1}{4};$$

$$a_3 = \frac{1}{6} a_2 + \frac{1}{12} a_1 = \frac{a_2}{6} = \frac{1}{24};$$

$$a_4 = \frac{2}{8} a_3 + \frac{1}{24} a_2 = \frac{1}{4} \cdot \frac{1}{24} + \frac{1}{24} \cdot \frac{1}{4}$$

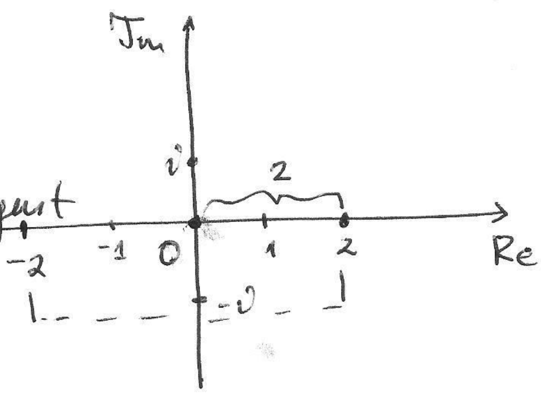
$$a_4 = \left(\frac{1}{4} + \frac{1}{4}\right) \frac{1}{24} = \frac{1}{48}.$$

$$\Rightarrow \boxed{y(x) = 1 + \frac{1}{4} x^2 + \frac{1}{24} x^3 + \frac{1}{48} x^4 + \dots}$$

(b) Standard form: $y'' + \frac{1}{x-2} y = 0$

$x = 2$ is the unique singular point.

$R = 2$ is the minimum radius of convergence $\Rightarrow$ the power series is convergent (represents $y(x)$) for all $\boxed{x : |x| < 2.}$



Solution 2 of (a): By using implicit differentiation:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots$$

$y(0) = 1$, $y'(0) = 0$.

$$(x-2)\,y''(x) + y(x) = 0 \,\Big|_{x=0} \Rightarrow (0-2)\,y''(0) + y(0) = 0$$

$$\Rightarrow y''(0) = \frac{1}{2}\,y(0) = \frac{1}{2}.$$

$$(x-2)\,y^{(3)}(x) + y''(x) + y'(x) = 0 \,\Big|_{x=0} \Rightarrow (0-2)\,y^{(3)}(0) + y''(0) + y'(0) = 0$$

$$\Rightarrow y^{(3)}(0) = \frac{1}{2}\,y''(0) = \frac{1}{4}.$$

$$(x-2)\,y^{(4)}(x) + y^{(3)}(x) + y^{(3)}(x) + y''(x) = 0 \,\Big|_{x=0}$$

$$-2\,y^{(4)}(0) + 2\,y^{(3)}(0) + y''(0) = 0$$

$$y^{(4)}(0) = \frac{1}{2}\left(2\,y^{(3)}(0) + y''(0)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$$

$$y(x) = 1 + \frac{0}{1!}x + \frac{1}{2\cdot(2!)}x^2 + \frac{1}{4(3!)}x^3 + \frac{1}{2(4!)}x^4 + \cdots$$

$$\boxed{y(x) = 1 + \frac{1}{4}x^2 + \frac{1}{24}x^3 + \frac{1}{48}x^4 + \cdots}$$