FINAL EXAM
PHYS 2332 (MODERN PHYSICS II)
DATE/TIME: April 8, 2011 (9:00 a.m. - 12:00 noon)
PLACE: ATAC 1006
Only non-programmable calculators are allowed.

Name: ___________________________________

ID: ___________________________________

Please read the following instructions:

- This midterm has 14 pages. Make sure none are missing.
- There are two sections. In section 1 you must do 4 out of 5 questions. In section 2 you must do 5 out of 6 questions.
- You must **show all works**, in the provided blank space below each question.
- Write your name and student ID in the provided space above.
- There is a two-page equation sheet on page 13 and 14. You may tear the equation sheet from the midterm.

Read questions carefully before attempting the solutions.

SECTION 1 Do 4 out of 5 questions. If you do all five, your final marks will be based on the best four marks.

Problem 1) The Spin-Orbit Coupling Effects splits the $3P \to 3S$ ($n=3, \ell=1 \to n=3, \ell=0$) transition of sodium into two lines: 589.0 nm corresponding to $3P_{3/2} \to 3S_{1/2}$ ($n=3, \ell=1, j=3/2 \to n=3, \ell=0, j=1/2$) transition; 589.6 nm corresponding to $3P_{1/2} \to 3S_{1/2}$ ($n=3, \ell=1, j=1/2 \to n=3, \ell=0, j=1/2$) transition.

A) Briefly explain how fine structure energy arises in atoms.
The outer valence electron of an atom sees the positively charge nucleus "orbiting" it. This give rises to an "internal" magnetic field B_{eff}, which is proportional to the orbital angular momentum of the atom $\vec{B}_{eff} \propto \vec{L}$, in direction and magnitude. This internal seen by the electron interacts with the elctron's spin giving an energy $E_{fs} = -\vec{\mu}_s \bullet \vec{B}_{eff}$, with $\vec{\mu}_s = -e\vec{S}/m_e$, the magnetic moment due to the electron's spin. It is this energy term that partially splits the degenerate electronic state of atoms giving a fine structure

B) Use these **wavelengths** to calculate the **effective magnetic field**, B_{eff}, experienced by the outer electron in the sodium atom as a result of its orbital motion.
Hint: When spin-orbit coupling is taken into account the energy of an electron in the n state is $E_{n,up} = E_n^0 - \mu_B B_{eff}$, for spin up, and $E_{n,down} = E_n^0 + \mu_B B_{eff}$ for spin down, where E_n^0 is the energy of the atom, without fine structure in the n electronic state, and $\mu_B = 5.788 \times 10^{-5} eV/T$. Assume E_n^0 does not depend on ℓ or m_ℓ.

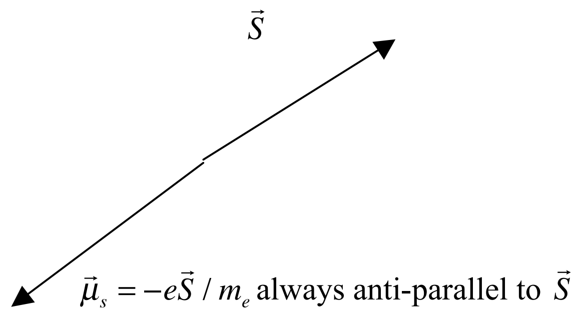
$$\Delta E = E_{n,down} - E_{n,up} = 2\mu_B B_{eff}$$

Energy of difference in photon energy $\Delta E = \frac{hc}{\lambda_2} - \frac{hc}{\lambda_1} \approx \frac{hc}{\lambda_{av}^2}\Delta\lambda$, with $\lambda_{av} = 5.893 \times 10^{-7} m$, and $\Delta\lambda = 0.6nm$, and $\Delta E = \frac{hc}{\lambda_{av}^2}\Delta\lambda = \frac{1239.8 eV \bullet nm}{(589.3nm)^2} \times 0.6nm = 2.14 \times 10^{-3} eV$

$$\Delta E = 2\mu_B B_{eff} \to B_{eff} = \Delta E / 2\mu_B = 2.14 \times 10^{-3} eV / 2\left(5.788 \times 10^{-5} eV/T\right) = 18.5T$$

Problem 2) Consider a **free electron**.
(A) What are the possible spin states of a free electron? What are the possible values of the magnetic moment due to the spin of the free electron? Draw a diagram showing the direction of the magnetic moment relative to the spin.

The spin $\vec{S}$ has magnitude $|\vec{S}| = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\hbar = \sqrt{\frac{3}{4}}\hbar$ and z-component $S_z = m_s\hbar$, with $m_s = \pm 1/2$, $S_z = \pm\hbar/2$. For spin magnetic moment $\vec{\mu}_s = -\frac{e}{m}\vec{S}$, its magnitude is $\mu_s = \sqrt{3/4}e\hbar/m_e$ and z-component $\mu_{s,z} = \mp e\hbar/2m_e = \mp\mu_B = \mp 5.788 \times 10^{-5} eV/T$.



(B) Suppose the free electron is in a magnetic field $\vec{B} = B_0\hat{k}$, where $B_0 = 1.2T$. What are its **possible** energies? The Bohr magneton $\mu_B = \dfrac{e\hbar}{2m} = 5.788\times10^{-5} eV/T$ may be useful. Express your answer in eV.

Use $E = -\vec{\mu}_s \bullet \vec{B}$, $E = -\vec{\mu}_s \bullet \vec{B} = -\left(-e\vec{S}/m_e\right) \bullet \vec{B} = \left(e/m_e\right)\left(\vec{S}\bullet\vec{B}\right) = \pm\dfrac{e\hbar}{2m_e}B_0 = \pm\mu_B B_0$

$E = \pm 5.788\times10^{-5} eV/T \times 1.2T = \pm 6.95\times10^{-5} eV$

(C) Briefly explain what is the **spin** of an electron.

The spin is an **intrinsic** quantum states of an electron, or other fermions or of bosons. Mathematically the spin obeys the same laws as for orbital angular momentum of atoms. In fact the spin interact with an applied (or internal) magnetic field in the same way as orbital angular momentum. The spin is essential for a complete explanation of the fine structure of atom and molecules, as well as for the anomalous Zeeman effects.

Problem 3) The fine structure of hydrogen If relativistic effects and spin-orbit coupling are taken into account, the energy level of a hydrogen atom, even without any external perturbation, is

$$E_{n,j} = -\frac{13.6eV}{n^2} - \frac{13.6eV}{n^4}\alpha^2\left[\frac{n}{j+1/2} - \frac{3}{4}\right],$$

where $n = 1,2,3...$, $j = \ell + 1/2, \ell - 1/2$, for $\ell > 0$, and $j = 1/2$ for $\ell = 0$. $\alpha = 1/137$ is the fine structure constant. The energy now depends on not just the principal quantum number, n, but also the total angular momentum quantum number, *j*, which is related to the orbital quantum number ℓ.

(A) Calculate the energy (in eV and in terms of α) of hydrogen up to the n = 3 level.

For n = 1, $\ell = 0$ (1s) and $j = 1/2$

$$E_{1,1/2} = -\frac{13.6eV}{1^2} - \frac{13.6eV}{1^4}\alpha^2\left[\frac{1}{1/2+1/2} - \frac{3}{4}\right] = -13.6eV - \left(3.4eV\right)\alpha^2$$

For n = 2, $\ell = 1$ (2p) and $j = 3/2$

$$E_{2,3/2} = -\frac{13.6eV}{2^2} - \frac{13.6eV}{2^4}\alpha^2\left[\frac{2}{3/2+1/2} - \frac{3}{4}\right] = -3.4eV - \left(0.2125eV\right)\alpha^2$$

For n = 2, $\ell = 1$ (2p) or $\ell = 0$ (2s), and $j = 1/2$

$$E_{2,1/2} = -\frac{13.6eV}{2^2} - \frac{13.6eV}{2^4}\alpha^2\left[\frac{2}{1/2+1/2} - \frac{3}{4}\right] = -3.4eV - (1.0625eV)\alpha^2$$

For n = 3, $\ell = 2$ (3d) and $j = 5/2$

$$E_{3,5/2} = -\frac{13.6eV}{3^2} - \frac{13.6eV}{3^4}\alpha^2\left[\frac{3}{5/2+1/2} - \frac{3}{4}\right] = -1.51eV - (0.042eV)\alpha^2$$

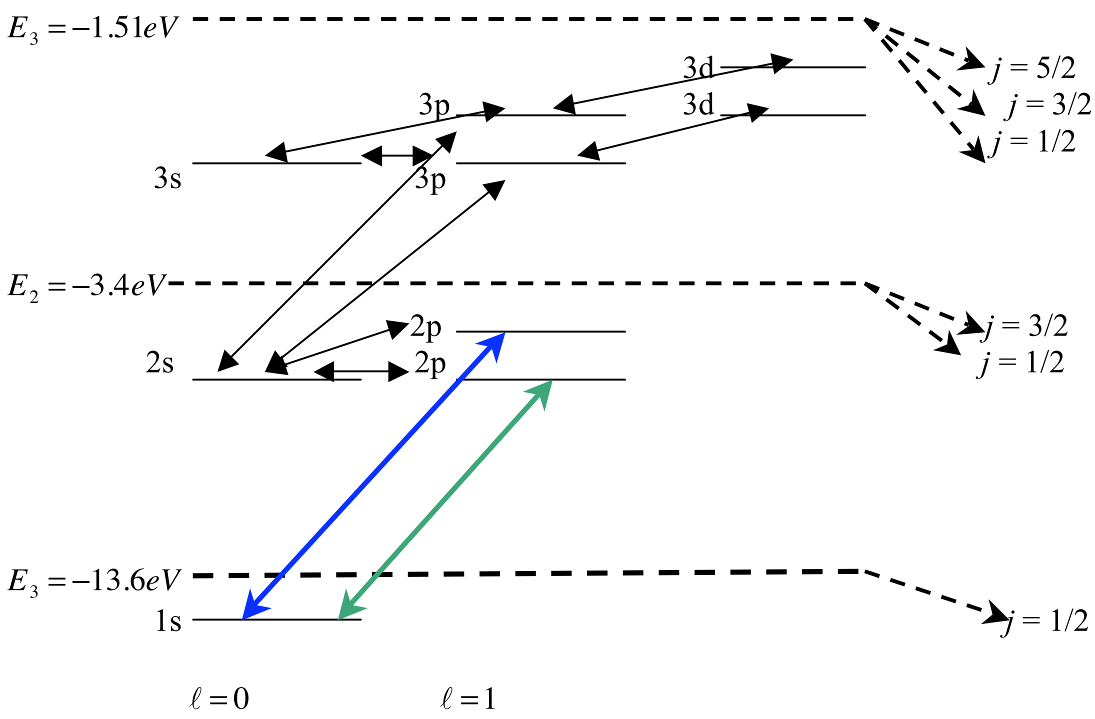
For n = 3, $\ell = 2$ (3d) or $\ell = 1$ (3p), and $j = 3/2$

$$E_{3,3/2} = -\frac{13.6eV}{3^2} - \frac{13.6eV}{3^4}\alpha^2\left[\frac{3}{3/2+1/2} - \frac{3}{4}\right] = -1.51eV - (0.08395eV)\alpha^2$$

For n = 3, $\ell = 1$ (3p) or $\ell = 0$ (3s), and $j = 1/2$

$$E_{3,1/2} = -\frac{13.6eV}{3^2} - \frac{13.6eV}{3^4}\alpha^2\left[\frac{3}{1/2+1/2} - \frac{3}{4}\right] = -1.51eV - (0.378eV)\alpha^2$$

(B) Using the result of part a), draw a **schematic energy-level diagram** of hydrogen up to the n = 3 level. Draw all possible dipole transitions (by photon absorption and emission) between hydrogen states. **Beware** of the selection rules.



selection rules: $\Delta\ell = \pm 1$, $\Delta j = 0, \pm 1$, $\Delta n = any$

Problem 4) Consider the rotational energy spectrum of HBr: $E_\ell = \ell(\ell+1)\frac{\hbar^2}{2I}$. The average number of particles in a state ℓ is given by the Maxwell-Boltzman Distribution: $n(E_\ell) = B(2\ell+1)\exp\left(-\beta\ell(\ell+1)\frac{\hbar^2}{2I}\right)$, where $\beta = \frac{1}{kT}$, and B is a constant that' s not important for this question.

A) At temperature of $T = 500K$ the peak intensity corresponds to the quantum number $\ell = 4$ (i.e. the most probable state is $\ell = 4$). Use the data m_H = 1amu, m_{Br} = 80 amu, and the equation $I = m^/R^2$ where $m^/$ is the reduced mass, estimate the bond length, R, of HBr.
Hint: minimize $n(\ell)$ then solve for *I*.

We simply minimize $n(E_\ell)$ w.r.t $\ell \rightarrow \frac{\partial n}{\partial \ell} = 0$

$$2B\exp\left(-\beta\ell(\ell+1)\frac{\hbar^2}{2I}\right) - B(2\ell+1)^2\beta\frac{\hbar^2}{2I}\exp\left(-\beta\ell(\ell+1)\frac{\hbar^2}{2I}\right) = 0$$

Using $\beta = \frac{1}{kT}$, $(2\ell+1)^2 = \frac{4kTI}{\hbar^2} \rightarrow \ell = \frac{1}{2}\sqrt{\frac{4kTI}{\hbar^2}} - \frac{1}{2}$

Solving for *I* gives $I = \frac{(2\ell+1)^2\hbar^2}{4k_BT} = \frac{(2\times4+1)^2(1.055\times10^{-34}J\bullet s)^2}{4\times1.381\times10^{-23}J/K\times500K} = 3.26\times10^{-47}kg\bullet m^2$

Reduced mass $m^/ = \frac{m_H m_{Br}}{m_H + m_{Br}} = \frac{(1.66\times10^{-27}kg)(1.33\times10^{-25}kg)}{1.66\times10^{-27}kg + 1.33\times10^{-25}kg} = 1.64\times10^{-27}kg.$

Using $I = m^/R^2 \rightarrow R = \sqrt{I/m^/} = \sqrt{3.26\times10^{-47}kg\bullet m^2/1.64\times10^{-27}kg} = 1.41\times10^{-10}m$.

B) Using the result of part A, calculate the most probable value of ℓ at $T = 800K$.

$$\text{T = 800 K}, J = \frac{1}{2}\sqrt{\frac{4(1.381\times10^{-23}J/K)(800K)(3.26\times10^{-47}kg\bullet m^2)}{(1.055\times10^{-34}J\bullet s)^2}} - \frac{1}{2} = 5.18\ 5.27$$

Rounding off gives $\ell = 5$.

Problem 5) It is known that the hydrogen molecule H_2 has a vibration absorption (emission) frequency of $\nu_0 = 1.32\times10^{14}Hz$.

(A) Model the H_2 molecule as two H atoms connected by a spring. Based on the data given calculate the spring constant k. Use m_H = 1amu.

Reduced mass $m^/ = \frac{m_H m_H}{m_H + m_H} = \frac{1}{2}m_H = 8.33\times10^{-28}kg.$

$$\omega = \sqrt{\frac{k}{\mu}} \rightarrow \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m^/}} \rightarrow k = 4\pi^2\nu^2m^/ = 4\pi^2(1.32\times10^{14}s^{-1})^2(8.33\times10^{-28}kg) = 573\frac{N}{m}$$

(B) Now consider a deuterium molecule D_2, where D is a heavy hydrogen with nucleus of one proton and one neutron with a mass m_D = 2amu. Use the data of part A to calculate the vibration frequency of this molecule.
For D_2,

$$m^{/} = \frac{m_D m_D}{m_D + m_D} = \frac{1}{2} m_D = 1.67 \times 10^{-27} kg.$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m^{/}}} = \frac{1}{2\pi}\sqrt{\frac{573N/m}{1.67 \times 10^{-27} kg}} = 9.3 \times 10^{13} Hz$$

(C) Briefly explain a "chemistry" assumption that you must make in order to do part B.
The assumption that the nuclear mass does not affect the chemical properties of the system means that we can use the same force constant k = 573 N/m.

<u>SECTION 2</u> Do 5 out of 6 questions. If you do all six, your final marks will be based on the best five marks.

Problem 6) The free electron density of cooper is $N/V = 8.45 \times 10^{28}$ electron/m^3 and its conductivity is $\sigma = 5.8 \times 10^{7} \Omega^{-1} \bullet m^{-1}$.

(A) Calculate the Fermi energy, ε_F.

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3}{8\pi}n\right)^{2/3} = \frac{\left(6.626 \times 10^{-34} J \bullet s\right)^2}{2\left(9.1 \times 10^{-31} kg\right)}\left(\frac{3}{8\pi}\left(8.45 \times 10^{28} m^{-3}\right)\right)^{2/3} = 1.126 \times 10^{-18} J = 7.04 eV$$

(B) Calculate the Fermi speed, u_F.

$$u_F = \sqrt{\frac{2\varepsilon_F}{m}} = \sqrt{\frac{2 \times 1.126 \times 10^{-18} J}{9.1 \times 10^{-31} kg}} = 1.57 \times 10^{6} \frac{m}{s}$$

(C) Calculate the mean-free path ℓ (the average distance traveled between collisions)
Start with the formula $\sigma = \frac{ne^2\ell}{mu_F} \rightarrow \ell = \frac{mu_F\sigma}{ne^2}$

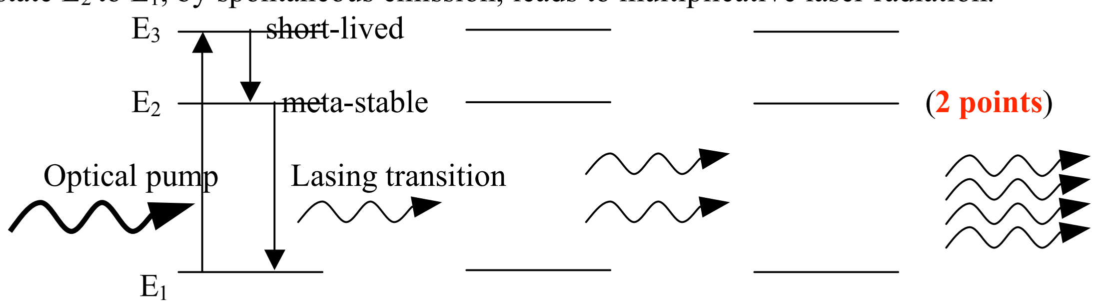
$$\ell = \frac{\left(9.1 \times 10^{-31} kg\right)\left(1.57 \times 10^{6} m/s\right)\left(5.8 \times 10^{7} \Omega^{-1} m^{-1}\right)}{\left(8.45 \times 10^{28} m^{-3}\right)\left(1.6 \times 10^{-19} C\right)^2} = 3.8 \times 10^{-8} m$$

Problem 7) Lasers
a) Briefly explain the difference between radiation from a laser and radiation from everyday material.
Usually radiations from materials are coherent meaning that they have a wide range of frequency (wavelength), have random phase and polarization, and are in random direction. Radiations from laser are coherent meaning that they are monochromatic (have a narrow range of frequency) have the same phase and polarization, and are in one direction. (**1 point**)

b) Briefly explain the meaning of photon absorption, stimulated emission, and spontaneous emission.
Materials can absorb a photon (EM radiation) and make a transition from a low-energy state to a high-energy state. Materials (i.e. atoms and molecules) can make a transition from a high-energy state to a low-energy state in two ways. They can decay spontaneously emitting a photon of equal energy to the energy difference between the states. In stimulated emission the transition can be stimulated by another photon resulted in an emitted photon of the same frequency, phase, and direction. (**1 point**)

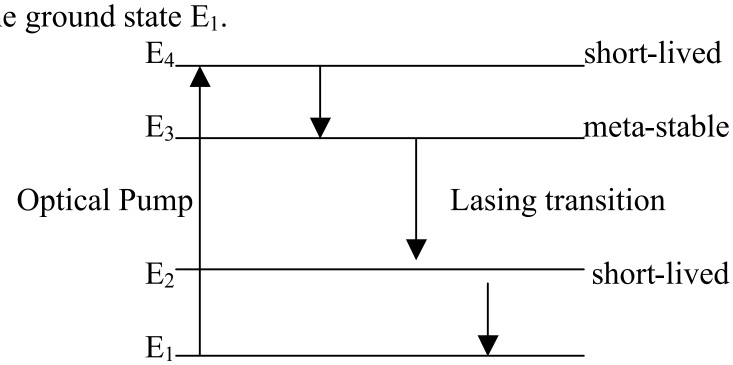
c) Briefly explain with a diagram how stimulated emission can be used to make a three-level and four-level laser.
To produce a laser there must be a population inversion of atomic (or molecular state), where there is a large number of atoms in a specific excited state. If one excited atom emits a photon by spontaneous emission. It can induce, by stimulated emission, another emitted photon from another atom. The two photons can induced the emission of two more photons, and the process multiplies. The emitted will be coherent. (**2 point**)

The diagram below explains how a three-level laser works. It begins with an optical pump that excited the atoms or molecules from a ground-energy state E_1 to short-lived high-energy state E_3 that quickly decay to a lower energy, creating a population inversion of meta-stable excited energy state E_2. An initial lasing transition from state E_2 to E_1, by spontaneous emission, leads to multiplicative laser radiation.



(**2 points**)

A four-level laser is shown below. Here the optical pump excites the atoms and molecules from the ground state E_1 to the short-lived E_4 state that quickly decay to the meta-stable E_3 excited state. This is followed by a lasing transition from the E_3 to E_2 that produces the laser radiation. Atoms in the short-lived E_2 state quickly decay to the ground state E_1.



(**2 points**)

d) What are the disadvantages of a three-level laser, and how is this fixed by a four-level laser?
In the three-level laser the laser radiation can be reabsorbed by an atom that has already decay to the ground state E_1, leading to a loss in intensity. In a four-level laser the excited state E_2 quickly decay to the ground state reducing the probability of absorption of laser photons. (**2 points**)

Problem 8) Semiconductors

(A) Using the band theory of solids explain the differences in the electrical conductivity behavior of conductors, insulators, and semiconductors. For full marks draw energy band diagram to illustrate your answer.

Done in class

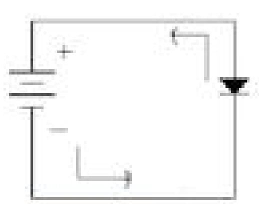
(B) Explain the differences between an n-type and a p-type semiconductor.

Usually pure semiconductors such as Germanium and Silicon has filled valence band gap separated from the conducting band by a gap of about 1eV. In n-type semiconductor, Silicon or Germanium is doped by material (P or As) with excess electrons, called the donor band, with energy close to the valence band, reducing the effective band gap. In p-type semiconductor, Silicon or Germanium is doped by material (B or Al) with excess vacancy (holes), called the acceptor band, with energy close to the conduction band, reducing the effective band gap.

(C) Germanium has atomic number $Z = 32$. Germanium is doped with Aluminum (Z = 13), would the resulting semiconductor be an n-type or an p-type. Briefly explain your answer.

Germanium has configuration $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^2$, with four valence electrons $4s^2 4p^2$. Aluminum has $1s^2 2s^2 2p^6 3s^2 3p^1$, with three valence electrons $3s^2 3p^1$, and introduces a vacancy to germanium. Hence this is a p type semiconductor.

(D) A diode has a reverse bias current of $I_0 = 2.0 \times 10^{-6} A$. Draw a circuit diagram (with appropriate symbol for a diode) when a forward voltage of $V = 3Volts$ is applied to the diode. Calculate current that flow to the circuit. Assume that the rest of the circuit has negligible, but nonzero, resistance.



LHS diagram is a forward bias diode

$$I = I_0\left(\exp\left(\frac{eV}{kT}\right) - 1\right) = 2.0 \times 10^{-6} A\left(\exp\left(\frac{1.6 \times 10^{-19} C \times 3V}{1.381 \times 10^{-23} J/K \times 300K}\right) - 1\right) = 4.1 \times 10^{44} A$$

Problem 9) Elementary Particle Physics Below is a diagram showing the creation of an electron-positron pair from a gamma ray photon. The reaction requires the presence of the nucleus to conserve energy and momentum.

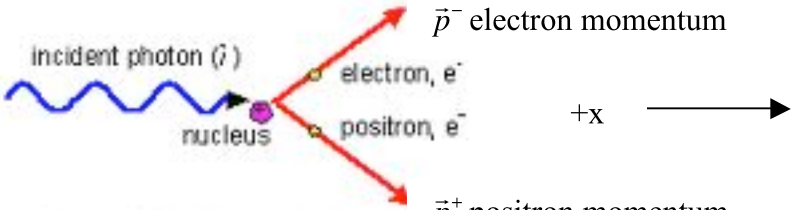


Figure 10.2 - Pair production.

A) To show that a nucleus is required for this reaction, write down the equation of conservation of energy. Students should note that the momentum of a photon is $h\nu / c$, where ν is the frequency of the gamma ray. For the electron and positron use the relativistic value of the energy $E = \sqrt{p^2c^2 + \left(mc^2\right)^2}$. Now write down the equation of conservation of momentum in the x-axis (horizontal). Compare the two equations carefully, and briefly (or show mathematically) why they cannot be satisfied simultaneously.

Let $E^+, \vec{p}^+$ be the energy and momentum of the positron, $E^-, \vec{p}^-$ be the energy and momentum of the electron, $h\nu, h\nu / c$ the energy and momentum of the photon.
Momentum conservation (y) photon $0 = p^- \sin\theta - p^+ \sin\theta$, electron + positron
Momentum conservation (x) photon $h\nu / c = p^- \cos\theta + p^+ \cos\theta$ electron + positron
Using pure math logic for above gives $h\nu \le p^-c + p^+c$
Energy Conservation: $h\nu = \sqrt{p^{+2}c^2 + \left(m_{e^+}c^2\right)^2} + \sqrt{p^{-2}c^2 + \left(m_{e^-}c^2\right)^2}$
Using pure math logic gives $h\nu > p^-c + p^+c$, which clearly contradicts the above inequalities, which shows that momentum and energy cannot be conserved simultaneously without the nucleus.

B) Complete the following reactions: $\mu^- + p \to n + ?$; $n + p \to \Sigma^0 + n + ?$. Briefly justify your answers?

$\mu^- + p \to n + \nu_\mu$, muon neutrino ν_μ has muon lepton number $L_\mu = 1$ needed to balance the $L_\mu = 1$ of μ^-

$n + p \to \Sigma^0 + n + K^+$, K^+ balances the charge of the proton on LHS, and also has strangeness number +1 that cancels the -1 strangeness of Σ^0 to equate with the zero strangeness of LHS. Also since K^+ is a meson, the baryon number of 2 on LHS is conserved.

C) Determine the quark composition of p, π^- and π^+.
p has baryon number of 1 so it must be composed of three quarks. It has zero charmed and strangeness number. For the charge to add up to + e, its composition must be *uud*.
π^- is a meson (baryon number 0) so it must be made of a quark/anti-quark pair. It has no strangeness or charm. To have charge –e, it must be composed of $\bar{u}d$.
π^+ is $u\bar{d}$ since it is the anti-particle of π^-.

D) Quarks are spin ½ particles. The Ξ^0 baryon has a quark composition ssu. Is this composition consistent with the Pauli exclusion principle? Why or why not?
All quarks are spin ½ particles, so it appears that the two s quark disobey the Pauli exclusion principle. However the existence quantum color states means that the ssu composition is valid as long as the two s quarks are in different color states.

Problem 10) White dwarf and neutron stars A dense star has a mass of 1.5 solar mass ($M_{Sun} = 2\times10^{30}\,kg$), and a radius of 12 km.

(A) Calculate the mass density of the star.

$$\rho = M/V = 1.5\times2\times10^{30}\,kg \div \left((4/3)\pi\times(12000m)^3\right) = 4.14\times10^{17}\,kg/m^3$$

(B) Assume that the star is a white dwarf, made up of ionized helium (nucleus of charge +2e and two free electrons). Noting that the helium nucleus is made up of two protons and two neutrons giving a nuclear mass of $m \approx 4amu$, calculate the **number density** of **helium ions** and **free electrons**. Hence calculate the **Fermi energy**, and the **pressure** of this **fermion** system. **NOTE:** Assume the ideal gas relation $PV = 2U/3$ is valid, where U is the total internal energy of the gas. For the calculation, neglect the pressure arising due to gravity and EM forces.

Number density of alpha particles

$$\left(\frac{N}{V}\right)_{alpha} = \frac{\rho}{M_{alpha}} = \frac{4.14\times10^{17}\,kg/m^3}{4\times1.67\times10^{-27}\,kg} = 6.2\times10^{43}\,m^{-3}$$

Since there are two electrons per alpha particles the electron density is $\left(\frac{N}{N}\right)_{electron} = 2\times\left(\frac{N}{V}\right)_{alpha} = 12.4\times10^{43}\,m^{-3}$. This gives the Fermi energy of

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3} = \frac{\left(6.626\times10^{-34}\,J\bullet s\right)^2}{2\times9.1\times10^{-31}\,kg}\left(\frac{3}{8\pi}\times12.4\times10^{43}\,m^{-3}\right)^{2/3} = 1.44\times10^{-8}\,J$$

Now use $U = 3N\varepsilon_F/5$ and

$$PV = 2U/3 \rightarrow P = \frac{2}{5}\frac{N}{V}\varepsilon_F = \frac{2}{5}12.4\times10^{43}\,m^{-3}\times1.44\times10^{-8}\,J = 7.14\times10^{35}\,Pa.$$

Note that the electrons comprise the degenerate Fermi gas.

(C) Assume instead that this is a neutron star made up entirely of neutrons, each with mass m_n = 1amu. Calculate the **Fermi energy** and **pressure** of the fermion system. Compare the answers of part B and C, and comment.

Here it is neutrons that comprise the Fermi gas. Using data from part B, the neutron number density is $\left(\frac{N}{N}\right)_{neutron} = 4\left(\frac{N}{V}\right)_{alpha} = 24.8\times10^{43}\,m^{-3}$

$$\varepsilon_F = \frac{h^2}{2m_n}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3} = \frac{\left(6.626\times10^{-34}\,J\bullet s\right)^2}{2\times1.67\times10^{-27}\,kg}\left(\frac{3}{8\pi}\times24.8\times10^{43}\,m^{-3}\right)^{2/3} = 1.26\times10^{-11}\,J$$

Note that I used the mass of the neutron $m_n = 1.67\times10^{-27}\,kg$.

$$P = \frac{2}{5}\frac{N}{V}\varepsilon_F = \frac{2}{5}24.8\times10^{43}\,m^{-3}\times1.26\times10^{-11}\,J = 6.25\times10^{32}\,Pa$$

The pressure in the neutron star is lower.

Problem 11) The Maxwell-Boltzman **speed** distribution is $n(v) = Av^2 \exp\left(-\frac{mv^2}{2kT}\right)$, $A = 4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2}$.

(A) Show that the root-mean-square-speed is $v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}}$, where m is the mass of the atom, $\langle v^2 \rangle = \frac{1}{N}\int_0^\infty v^2 n(v)\,dv$. Integration table in back is useful.

$$\langle v^2 \rangle = \frac{1}{N}\int_0^\infty v^2 n(v)dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^4 \exp\left(-\frac{mv^2}{2kT}\right)dv \text{ (**1 point**)}$$

Using $\int_0^\infty r^4 e^{-ar^2} = \frac{3}{8}\frac{\sqrt{\pi}}{a^{5/2}} \rightarrow \langle v^2 \rangle = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{3\sqrt{\pi}}{8}\left(\frac{2kT}{m}\right)^{5/2} = \frac{3kT}{m}$ (**1.5 point**)

This gives $v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}}$ (**0.5 point**)

(B) Show that the most probable speed of an ideal gas atom is $v_p = \sqrt{\frac{2kT}{m}}$, where m is the mass of the atom.

The most probable speed is found by the condition $\frac{\partial n}{\partial v} = 0$ (**1 point**)

$\frac{\partial n}{\partial v} = 2Av\exp\left(-\frac{mv^2}{2kT}\right) - 2A\frac{mv^3}{2kT}\exp\left(-\frac{mv^2}{2kT}\right) = 0 \rightarrow v^2 = \frac{2kT}{m}$ (**1.5 point**), giving the most probable speed $v_p = \sqrt{\frac{2kT}{m}}$ (**0.5 point**)

(C) Consider an ideal gas of Neon, Ne, with $m_{Ne} = 20\text{amu}$. Calculate the most probable speed of a Neon atom (of an ideal gas) at T = 300° K and at T = 1000° K. Explain, using physical reasons, why you expect v_p to be higher at T = 1000° K.

$m_{Ne} = 20\text{amu} \times 1.66\times10^{-27}\,kg/amu = 3.32\times10^{-26}\,kg$ (**0.5 point**)

T = 300° K, $v_p = \sqrt{\frac{2(1.381\times10^{-23}\,J/K)(300K)}{3.32\times10^{-26}\,kg}} = 499\,m/s$ (**0.5 point**)

T = 1000° K, $v_p = \sqrt{\frac{2(1.381\times10^{-23}\,J/K)(1000K)}{3.32\times10^{-26}\,kg}} = 912\,m/s$ (**0.5 point**)

At T = 1000° K, the atoms have more energy and consequently have higher speed. (**0.5 point**)

(D) Repeat the calculation of part (C) for an ideal gas of Argon, Ar, with $m_{Ar} = 40\text{amu}$ at T = 300° K and at T = 1000° K. Explain, using physical reasons, why you expect v_p to be lower for Ar than for Ne.

$m_{Ar} = 40\text{amu} \times 1.66\times10^{-27}\, kg/amu = 6.64\times10^{-26}\, kg$ (**0.5 point**)

T = 300° K, $v_p = \sqrt{\dfrac{2(1.381\times10^{-23} J/K)(300K)}{6.64\times10^{-26} kg}} = 353 m/s$ (**0.5 point**)

T = 1000° K, $v_p = \sqrt{\dfrac{2(1.381\times10^{-23} J/K)(1000K)}{6.64\times10^{-26} kg}} = 645 m/s$ (**0.5 point**)

The speed of the "heavier" argon is greater than neon. (**0.5 point**)

Useful Equations

Hydrogen Hydrogen in $n,\ \ell,\ m_\ell$ has the wavefunction $\psi_{n\ell m_\ell} = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$

Conserved Quantities: $E_n = -\dfrac{13.6eV}{n^2}$, $n = 1,2,3...$; $L = \sqrt{\ell(\ell+1)}\hbar, \ell = 0,1,2..,n-1$;

$L_z = m_\ell \hbar,\ m_\ell = 0,\pm1,\pm2,....\pm\ell$. Spin: $S = \dfrac{\sqrt{3}}{2}\hbar,\ S_z = m_s\hbar;\ m_s = +\dfrac{1}{2}(\text{up}),\ -\dfrac{1}{2}(\text{down})$

Magnetic moment due to a spin: $\vec{\mu}_s = -\dfrac{e}{m}\vec{S}$, energy in magnetic field $E = -\vec{\mu}_s \bullet \vec{B}$

Selection rule for electronic dipole transition in hydrogen and atoms $\Delta n =$ any, $\Delta\ell = \pm1$, $\Delta m_\ell = 0,\pm1$

Energy of a photon: $E = h\nu$ and $\lambda = c/\nu$, where $c = 2.998\times10^8 m/s$

Maxwell-Boltzman; speed dist. $F(v)dv = 4\pi N\left(\dfrac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\dfrac{mv^2}{2k_BT}\right)dv$; velocity dist. $g(v_x) = \left(\dfrac{m}{2\pi k_B T}\right)^{1/2} e^{-\frac{mv_x^2}{2k_BT}}$; root-mean-square speed $v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\dfrac{3k_BT}{m}}$

Ideal Gas $PV = NK_BT = 2U/3$; $U = 3Nk_BT/2$

Fermi-Dirac $f_{FD} = \dfrac{1}{e^{(\varepsilon-\mu)/k_BT}+1}$. **Bose-Einstein** $f_{BE} = \dfrac{1}{e^{(\varepsilon-\mu)/k_BT}-1}$.

Fermi energy $\varepsilon_F = \dfrac{h^2}{2m}\left(\dfrac{3}{8\pi}\dfrac{N}{V}\right)^{2/3}$; $T_F = \dfrac{\varepsilon_F}{k_B}$. $u_F = \sqrt{\dfrac{2\varepsilon_F}{m}}$; $U = 3N\varepsilon_F/5$

Rotational Levels: $E_\ell = \dfrac{L^2}{2I} = \dfrac{\ell(\ell+1)}{2I}\hbar^2, \ell = 0,1,2...$, where $I = \mu R^2$ and $\mu = \dfrac{m_1 m_2}{m_1+m_2}$. $\Delta E_{\ell+1\leftrightarrow J} = E_{\ell+1} - E_\ell = \dfrac{(\ell+1)\hbar^2}{I}$. Absorption and Emission of photons: i) selection rule $\Delta\ell = \pm1$; ii) $h\nu = \Delta E_{\ell\leftrightarrow\ell+1}$.

Vibrational Levels: $E_{vibr} = (n+1/2)\hbar\omega$, n = 0,1,2...,where $\omega = \sqrt{\dfrac{k}{\mu}}$,and $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.

$\Delta E_{n+1\leftrightarrow n} = E_{n+1} - E_n = \hbar\omega$. Absorption and Emission of photons: i) selection rule $\Delta n = \pm 1$; ii) $h\nu = \Delta E_{n+1\leftrightarrow n}$. **Dissociation Energy (U_0):** $E_{\text{vibr}} = (n+1/2)\hbar\omega - U_0$

Trigonometric relation $\cos^2 x + \sin^2 x = 1$; $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$; $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$; $\sin 2A = 2\sin A \cos A; \cos 2A = \cos^2 A - \sin^2 A$; $\sin^2 \dfrac{A}{2} = \dfrac{1-\cos A}{2}; \cos^2 \dfrac{A}{2} = \dfrac{1+\cos A}{2}$;

Integrals $\int_0^\infty e^{-ra} = \dfrac{1}{a}$; $\int_0^\infty re^{-ra} = \dfrac{1}{a^2}$; $\int_0^\infty r^2 e^{-ra} = 2\dfrac{1}{a^3}$; $\int_0^\infty r^3 e^{-ra} = 6\dfrac{1}{a^4}$; $\int_0^\infty e^{-ar^2} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}$; $\int_0^\infty r^2 e^{-ar^2} = \dfrac{1}{4}\dfrac{\sqrt{\pi}}{a^{3/2}}$; $\int_0^\infty r^4 e^{-ar^2} = \dfrac{3}{8}\dfrac{\sqrt{\pi}}{a^{5/2}}$.

Solid-State Device: diode current $I = I_0\left(\exp\left(\dfrac{eV}{kT}\right) - 1\right)$; conductivity $\sigma = \dfrac{ne^2\ell}{mu_F}$ of resistor, resitivity $\rho = 1/\sigma$, resistance $R = L\rho/A$

Useful constants: Atomic mass unit (amu) 1 amu$=1.66\times10^{-27}\,kg$;
electron rest mass $m_e = 9.11\times10^{-31}\,kg$; proton rest mass $m_p = 1.66\times10^{-27}\,kg$;
Electron charge $e = 1.6\times10^{-19}\,C$;electron volt 1 eV $= 1.6\times10^{-19}\,J$;
Planck constant $h = 6.62\times10^{-34}\,J\bullet s = 4.136\times10^{-15}\,eV\bullet s$;
$\hbar = 1.055\times10^{-34}\,J\bullet s = 6.582\times10^{-16}\,eV\bullet s$;
Boltzman's constant $k = 1.381\times10^{-23}\,J/K = 8.617\times10^{-5}\,eV/K$.
$\varepsilon_0 = 8.854\times10^{-12}\,C^2/N\bullet m^2$

Particle Physics

Lepton: lepton's number is – 1 for anti-particle. Tao leptons are not included here.

name	symbol	Anti-particle	L_e	L_μ
electron	e^-	e^+	1	0
e-neutrino	ν_e	$\bar{\nu}_e$	1	0
muon	μ^-	μ^+	0	1
$\mu-$neutrino	ν_μ	$\bar{\nu}_\mu$	0	1

Hadrons: baryon is made of three quarks; mesons is made of a quark-anti-quark pair.

name	symbol	Anti-particle	Baryon	Strangeness
proton	p	$\bar{p}$	1	0
neutron	n	$\bar{n}$	1	0
meson	π^-	π^+	0	0
Sigma	Σ^0	$\bar{\Sigma}^0$	1	-1
	Σ^+	$\bar{\Sigma}^-$	1	-1
Kaon	K^+	K^-	0	1
	K^0	$\bar{K}^0$	0	1

Quarks: corresponding anti-quark ($\overline{q}$) of a quark (q) has opposite charge, baryon number, and strangeness number.

name	symbol	charge	Baryon	Strangeness
up	u	2e/3	1/3	0
down	d	-e/3	1/3	0
Strange	s	-e/3	1/3	-1
Charmed	c	2e/3	1/3	0