MAT 2322C - Winter 2015
Solutions to Midterm I

1. If $f(x, y)$ is a differentiable function such that $\vec{\nabla} f(1,2) = 2\vec{i} + \vec{j}$, only one of the following curves can be the level curve for f through the point $(1, 2)$. Which one?

A. $y = \frac{3}{2} + \frac{x}{2}$
B. $y = 1 + e^{x-1}$
C. $x = 1$
D. $y = 2e^{x-1}$
E. $y = \dfrac{2}{x}$
F. $y = 2 + (x-1)^2$

We know that at any point in the plane, the level curve through the point is always perpendicular to the gradient vector at the point. This means that the slope (i.e. derivative $\dfrac{dy}{dx}$) is perpendicular to the slope of the gradient vector (rise/run). Here, the gradient vector at the point (1,2) is $2\vec{i} + \vec{j} = (2, 1)$, so its slope is $1/2$. Recall that if a line has slope m, then its perpendicular has slope $-1/m$. Therefore the level curve must have slope -2 at the point (1,2). All the listed curves pass through the point (1,2), and checking the slopes, we see that only (E) has derivative $\dfrac{dy}{dx}$ at $x = 1$ equal to -2.

2. Let f be given by $f(x, y, z) = x^2 + 3y^2 - 3z^3$. Which of the following describes the tangent plane to the surface $f(x, y, z) = 4$ at the point $(2, 1, 1)$?

The formula for the tangent plane is given on the last line of page 796 of the text; a concise form of it is as follows: at a point $\vec{\boldsymbol{x}}_0 = (x_0, y_0, z_0)$, we first calculate the gradient $\boldsymbol{\nabla} f(\vec{\boldsymbol{x}}_0) = \boldsymbol{\nabla} f(x_0, y_0, z_0)$ of f at the point; the tangent plane is given by the following equation in $\vec{\boldsymbol{x}} = (x, y, z)$:

$$\boldsymbol{\nabla} f(\vec{\boldsymbol{x}}_0) \cdot (\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_0) = 0$$

or equivalently

$$\boldsymbol{\nabla} f(\vec{\boldsymbol{x}}_0) \cdot \vec{\boldsymbol{x}} = \boldsymbol{\nabla} f(\vec{\boldsymbol{x}}_0) \cdot \vec{\boldsymbol{x}}_0.$$

In this problem, $\boldsymbol{\nabla} f = (2x, 6y, -9z^2)$, and $\vec{\boldsymbol{x}}_0 = (2, 1, 1)$, so $\boldsymbol{\nabla} f(\vec{\boldsymbol{x}}_0) = (4, 6, -9)$, and $\boldsymbol{\nabla} f(\vec{\boldsymbol{x}}_0) \cdot \vec{\boldsymbol{x}}_0 = 4(2) + 6(1) + (-9)(1) = 5$. Hence the equation defining the tangent plane is $(4, 6, -9) \cdot (x, y, z) = 5$, or equivalently $4x + 6y - 9z = 5$.

3. (a) Suppose that $z = f(x, y)$, and x and y are functions of variables u and v: $x = x(u, v)$, $y = y(u, v)$. The chain rule gives a formula for $\dfrac{\partial z}{\partial v}$. Write down this formula.

As seen in class,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v};$$

if z depended on three variables instead of just the x and y of this problem, then we would add three terms.

(b) Suppose that $z = xe^{y/x}$, and that $x = \cos(u + v)$, $y = u^2 + uv$. Use the result from part (a) to find a formula for $\dfrac{\partial z}{\partial v}$ in terms of u and v only. Do **not** try to simplify it.

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \left(e^{y/x} + xe^{y/x}(-y/x^2)\right)(-\sin(u+v)) + (xe^{y/x}(1/x))(u) \\
&= \left(e^{y/x} - \frac{y}{x}e^{y/x}\right)(-\sin(u+v)) + e^{y/x}u \\
&= \left(e^{(u^2+uv)/\cos(u+v)} - \frac{u^2+uv}{\cos(u+v)}e^{(u^2+uv)/\cos(u+v)}\right)(-\sin(u+v)) + e^{(u^2+uv)/\cos(u+v)}u,
\end{aligned}$$

where in the last line we switched every x and y to their expression in terms of u and v.

4. Find and classify the critical points of the function $f(x, y) = 4x - 3x^3 - 2xy^2$.

Critical points are the places where ∇f is either the zero vector or undefined. For this function it is always defined, so we need to solve two equations in two unknowns: $f_x = 0$ and $f_y = 0$. Explicitly,

$$\begin{aligned} 4 - 9x^2 - 2y^2 &= 0 \\ -4xy &= 0. \end{aligned}$$

The second equation is easier to start with, it holds exactly when at least one of x and y is zero. We treat these two cases separately.

Case 1, $x = 0$: Then the first equation gives $4 - 2y^2 = 0$, so $y = \pm\sqrt{2}$. This gives two points $(0, \sqrt{2})$ and $(0, -\sqrt{2})$ on the y-axis.

Case 2, $y = 0$: Then the first equation gives $4 - 9x^2 = 0$, so $x = \pm 2/3$. This gives two points $(2/3, 0)$ and $(-2/3, 0)$ on the x-axis.

So there are four critical points, listed above. To classify them, we calculate $D = f_{xx}f_{yy} - (f_{xy})^2$.

$$f_{xx} = -8x, \quad f_{yy} = -4x, \quad f_{xy} = -4y, \qquad \text{so } D = 32x^2 - 16y^2.$$

For the two points on the y-axis, $D = 0 - 16 * 2 < 0$ and so they are both saddle points.

For the two points on the x-axis, $D > 0$ and so they are either local max or mins. We check by evaluating f_{xx}:
Since $f_{xx}(2/3, 0) = -8 * (2/3) < 0$, the point (2/3,0) is a local maximum.
Since $f_{xx}(-2/3, 0) = -8 * (-2/3) > 0$, the point (-2/3,0) is a local minimum.

5. Find the global maximum and the global minimum of the function $f(x, y) = xy$ on the region

$$A = \{ (x, y) \in \mathbb{R}^2 \mid 4x^2 + 9y^2 \leq 32 \}.$$

The region A is a filled-in ellipse. We first look for possible maxima on the inside of the ellipse, and then for possible maxima on the curve of the ellipse itself; we will get a collection of points and compare the value of f at those points.

For the inside of the ellipse, we look for critical points: $f_x = y = 0$ and $f_y = x = 0$. The only point that satisfies both conditions is (0,0). (We could check separately whether it is a local max, and it turns out that it isn't; actually it's a saddle point for f; so there are nearby points with higher values.) The value of f at (0,0) is 0.

For the ellipse itself, it is the points on the curve $g(x, y) = 4x^2 + 9y^2 = 32$. The points on the curve with extreme values of f are determined using Lagrange multipliers:

$$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g, \quad g = c,$$

that is,

$$y = 8\lambda x, \quad x = 18\lambda y, \quad 4x^2 + 9y^2 = 32. \tag{$*$}$$

Plugging the second equation into the first gives $y = 8 * 18\lambda^2 y$, so either $y = 0$ or we can cancel y to get $\lambda^2 = 1/144$. We treat these two cases separately.

If $y = 0$, then the second equation tells us that x also equals 0, but then the third equation of (*) fails. So this case gives no points.

If $\lambda^2 = 1/144$, then $\lambda = \pm 1/12$. If $\lambda = 1/12$, then the first equation becomes $y = 2/3z$, and the third equation becomes $4x^2 + 9(2/3x)^2 = 8x^2 = 32$, which has two solutions, $x = \pm 2$. The formula $y = 2/3x$ then tells us that the corresponding points are $(2, 4/3)$ and $(-2, -4/3)$. At both these points, the value of f is 8/3.

Similarly, if λ is $-1/12$, we end up with the two points $(2, -4/3)$ and $(-2, 4/3)$. At both these points, the value of f is $-8/3$.

Comparing the values of f at the points we've computed (0 and $\pm 8/3$), we see that the maximum value of f on A is 8/3 and its minumum is $-8/3$.

6. Compute the following double integrals $\iint_R (3x^5 + 5xy^4)\,dA$, for each of the regions R described below:

(a) R is the rectangle $0 \le x \le 2$, $0 \le y \le 1$.

(b) R is the triangle in the plane with vertices at $(0,0)$, $(0,1)$ and $(1,1)$.

(a) We convert this to an iterated integral. We could put x or y on the outside, we'll pick x (they are equally easy in this case). So the desired integral is

$$\int_0^2 \int_0^1, (3x^5 + 5xy^4)\,dy\,dx = \int_0^2 \left.(3x^5 y + xy^5)\right|_0^1 dx$$
$$= \int_0^2 (3x^5 + x)\,dx = \left.\frac{1}{2}(x^6 + x^2)\right|_0^2 = \frac{1}{2}(64+4) = 34.$$

(b) This time it helps to draw a picture. Again, we could put x or y on the outside. We'll pick y as it is slightly easier (this way both lower limits equal 0). The lowest and highest values of y in the region R are 0 and 1, respectively. Next, we fix a y value (i.e. a horizontal line with that value for y) and determine which x values are allowed to stay within R. The lowest x value is 0. The highest x value is given by where the horizontal line bumps into the diagonal line $y = x$; solving for x, with this value of y, the value is $x = y$. So the integral is

$$\int_0^1 \int_0^y (3x^5 + 5xy^4)\,dx\,dy = \int_0^1 \left(\left. \frac{3}{6}x^6 + \frac{5}{2}x^2y^4 \right|_0^y \right) dy$$
$$= \int_0^1 \left(\frac{1}{2}y^6 + \frac{5}{2}y^6 \right) dy$$
$$= \int_0^1 3y^6\,dy = \left.\frac{3}{7}y^7\right|_0^1 = \frac{3}{7}.$$