Chapter 10: Dimension Theorems

From previous chapter:

the size of any LI set in $V \leq$ the size of any spanning set of V

we used it and deduced that

- any "linearly independent spanning" set of a vector space V (we called it a "basis" of V) has the same number of elements.
- The number of elements in a basis of $V = n = \dim(V)$.

we have also shown:

(1) If $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is a LI subset of V, then there are vectors $\vec{v}_{m+1}, \ldots, \vec{v}_n$ in V such that $\{\vec{v}_1, \ldots, \vec{v}_m, \vec{v}_{m+1}, \ldots, \vec{v}_n\}$ is a basis for V (that is, every linearly independent subset of V can be extended to a basis of V).

(2) Conversely, we saw that if $S = \{\vec{v}_1, \ldots, \vec{v}_\ell\}$ is a spanning set of V, then there is a subset of S which is a basis of V (that is, every spanning set can be reduced to a basis of V).

<u>Remark</u>: Every subspace of a finite-dimensional space has a finite basis.

In theory at least, we can always find a basis for any subspace: either start with a spanning set and cut it down, or else start with a LI subset and form larger and larger LI subsets.

<u>Theorem (10.1)</u> : (shortcut to deciding if a set is a basis)

suppose, we know that $\dim(V) = n$ & $n < \infty$. Then,

(1) Any LI set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of n vectors in V is a basis of V.
(Also, necessarily spans V).

(2) Any spanning set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V consisting of exactly n vectors is a basis of V (also, necessarily LI).

So, if we know $\dim(V)$, we can find a basis by either finding an LI $\boxed{\text{or}}$ a spanning set with the same number of elements.

<u>proof</u> :

(1) suppose $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is LI and that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ didn't span V. Then, we could find $\vec{v} \in V$, such that $\{\vec{v}, \vec{v}_1, \ldots, \vec{v}_n\}$ is LI, with $(n+1)$ LI elements in V, which contradicts the important inequality of last page, since $\dim(V) = n$ means that V can be spanned by n elements.

(2) suppose $V = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$, but that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is not LI. Using the same argument, we will reach a contradiction to the inequality. So, the spanning set must be LI and hence a basis. $\square$

<u>Example</u>: $\{(2,2,2), (7,1,-11)\}$ is a basis for the plane

$$V = \{(x,y,z) \mid 2x - 3y + z = 0\}.$$

V is a plane through origin, and both vectors lie on V:

$$2(2) - 3(2) + 2 = 0 \quad \& \quad 2(7) - 3(1) - 11 = 0$$

vectors $(2,2,2)$ & $(7,1,-11)$ are not multiple of each other, so they are LI and they form a basis for V (by theorem (10.1)).

<u>Example</u>: Enlarge $\{(2,2,2), (7,1,-11)\}$ to a basis in $\mathbb{R}^3$.

we know $\dim(\mathbb{R}^3) = 3$; so if we can find a $\vec{v} \in \mathbb{R}^3$, such that $\{\vec{v}, (2,2,2), (7,1,-11)\}$ is LI, we can imply that they form a basis of $\mathbb{R}^3$ by using the theorem (10.1).

$\{(2,2,2), (7,1,-11)\}$ is LI, therefore $\{\vec{v}, (2,2,2), (7,1,-11)\}$ is LI if & only if $\vec{v} \notin \operatorname{span}\{(2,2,2), (7,1,-11)\}$, for instance $\vec{v} = (1,0,0)$ would do that. Hence, $\{(1,0,0), (2,2,2), (7,1,-11)\}$ is a basis for $\mathbb{R}^3$.

⊛ <u>Theorem (10.2): Dimensions of subspaces</u>

suppose, $\dim(V) = n$ and that W is a subspace of V. Then,

(1) $0 \leqslant \dim(W) \leqslant \dim(V)$,

(2) $\dim(W) = \dim(V)$ if and only if $W = V$,

(3) $\dim(W) = 0$, if and only if $W = \{\vec{0}\}$.

proof :

(1) Start with a basis of W; it has $\dim(W)$ elements (≥ 0). Then, this is a LI set in $W \subset V$, so by the import inequality from page 58 of these notes, $\dim(W) \leq \dim(V)$.

(2) Suppose $\dim(W) = \dim(V) = n$. If $W \neq V$, then there is a $\vec{v} \in V$, such that $\vec{v} \notin W$. Then, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for W, we could deduce that $\{\vec{v}, \vec{v}_1, \ldots, \vec{v}_n\}$ is a LI set in V with $n+1$ elements, which contradicts the inequality for V. So, we must have $V \subset W$ (and we started with $W \subset V$, so $V = W$).

(3) See the remark at the end of page 57 of the notes.

$\square$

Examples :

(1) Any subspace of $\mathbb{R}^3$ has dimension $0, 1, 2, 3$ by the theorem (10.2)

They correspond to the zero space, lines, planes and $\mathbb{R}^3$ itself.

(2) Any 4-dimensional subspace of $M_{2\times2}(\mathbb{R})$ is all of $M_{2\times2}(\mathbb{R})$.

(3) Any 2-dimensional subspace of $U = \{(x,y,z) \mid 2x - 3y + z = 0\}$ is all of U.

Note : If U is a subspace of V and $\dim(U) = m$, then any subspace of V, which is contained in U is a subspace of U and so has dimension at most m.

Applications of bases:

* **Definition**: An "ordered basis" $\{\vec{v}_1, \dots, \vec{v}_n\}$ is the set $\{\vec{v}_1, \dots, \vec{v}_n\}$ with the given order of the elements.

* Theorem (10.3): (Coordinates)

  Suppose $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an ordered basis for a vector space V. Then, for every $\vec{v} \in V$, there are unique scalars $\{x_1, \dots, x_n \in \mathbb{R}$, such that

$$\vec{v} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n ,$$

and we call the n-tuple $(x_1, \dots, x_n)$ the coordinates of $\vec{v}$ relative to the ordered basis B.

Proof: Uniqueness: B spans V, so

$$\vec{v} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n , \quad \text{and}$$

$$\vec{v} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + \cdots + y_n \vec{v}_n .$$

so, $\vec{v} - \vec{v} = (x_1 - y_1)\vec{v}_1 + (x_2 - y_2)\vec{v}_2 + \cdots + (x_n - y_n)\vec{v}_n = \vec{0}$,

but B is LI, so $(x_i - y_i) = 0$, for all $i = 1, \dots, n$.

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We can use the idea of coordinates and ordered bases to "identify" n-dimensional vector spaces with $\mathbb{R}^n$:

$$V \longleftrightarrow \mathbb{R}^n$$

$$\vec{v} = x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n \longleftrightarrow (x_1, \dots, x_n) \in \mathbb{R}^n$$

<u>Examples</u>:

(1) $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$: ordered basis of $M_{2\times 2}(\mathbb{R})$ (in fact, "standard ordered basis"):

$$M_{2\times 2}(\mathbb{R}) \longleftrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow (a, b, c, d)$$

(2) $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$: ordered basis of SL_2.

then,

$$SL_2 \longleftrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longleftrightarrow (a, b, c)$$

(3) $B = \{1, x, x^2\}$: standard ordered basis of $\mathbb{P}_2$, then

$$\mathbb{P}_2 \longleftrightarrow \mathbb{R}^3$$

$$a + bx + cx^2 \longleftrightarrow (a, b, c)$$

$B' = \{x^2, x, 1\}$: ordered basis of $\mathbb{P}_2$, then

$$\mathbb{P}_2 \longleftrightarrow \mathbb{R}^3$$

$$ax^2 + bx + c \longleftrightarrow (c, b, a)$$

<u>Note</u>: The order of bases matters.