University of Ottawa - Department of Mathematics and Statistics
MAT 1322 D - Calculus II
Instructor: Petko Kitanov
October 14, 2015
Midterm Examination I
Version 1

Solutions

Blue

Name:...................................... Student Number:...........

Instructions :

- Please write your name and student number on the indicated area above.
- This is a closed book exam. It contains **6 questions**; there are 50 points in total.
- You can use non-programable and non-graphical calculators but no other aids are permitted.
- Clearly indicate the solution of each problem.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- If you need extra space, use the last two pages or the back of the pages.
- Time allowed: 80 minutes.

GOOD LUCK!

Student Number :____________________ Final Grade : ___________ out of 50

Question	1	2	3	4	5	6
Grade						

Question 1.

a) [4 pts] Evaluate the improper integral $\int_e^\infty \frac{1}{x(\ln x)^{\frac{3}{2}}}\,dx$

b) [4 pts] Use the Comparison Theorem to determine whether the integral $\int_1^\infty \frac{2x\cos^2 x}{\sqrt{9x^6+2x}}\,dx$ is convergent or divergent.

(a) $$\int_e^\infty \frac{dx}{x(\ln x)^{3/2}} = \lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^{3/2}}$$

$$\int \frac{dx}{x(\ln x)^{3/2}} = \int (\ln x)^{-3/2}\,d(\ln x) = -2\frac{1}{\sqrt{\ln x}}$$

$$\lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^{3/2}} = \lim_{t\to\infty}\left[-2\frac{1}{\sqrt{\ln x}}\right]_e^t = \lim_{t\to\infty}\frac{-2}{\sqrt{\ln t}} + \frac{2}{\sqrt{\ln e}}$$
$$= 0 + 2 = 2$$

(b) $$\frac{2x\cos^2 x}{\sqrt{9x^6+2x}} \underset{\cos^2 x \le 1}{\le} \frac{2x}{\sqrt{9x^6+2x}} \underset{x\ge 1}{\le} \frac{2x}{\sqrt{9x^6}} = \frac{2}{3x^2}$$

So, we have

$$\int_1^\infty \frac{2x\cos^2 x}{\sqrt{9x^6+2x}}\,dx \le \int_1^\infty \frac{2}{3x^2}\,dx$$

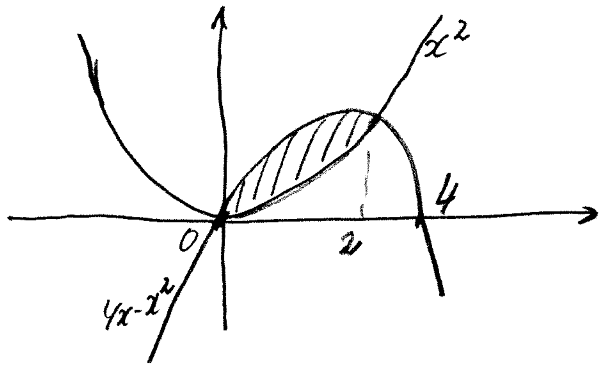
$$\underset{\text{convergent}}{\downarrow} \Longleftarrow \underset{\text{convergent}}{\downarrow}$$

Question 2. [10 pts]

a) Sketch the region enclosed by the curves $y = x^2$, $y = 4x - x^2$ and find its area.

b) Find the exact length of the curve $y = x^{\frac{3}{2}}$, $0 \le x \le 3$

(a)



$$x^2 = 4x - x^2$$
$$2x^2 - 4x = 0$$
$$x(x-2) = 0$$
$$x = 0, \quad x = 2$$

$$A = \int_0^2 [4x - x^2 - x^2]\,dx = \int_0^2 (4x - 2x^2)\,dx$$

$$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 8 - \frac{16}{3} = \frac{8}{3}$$

(b)

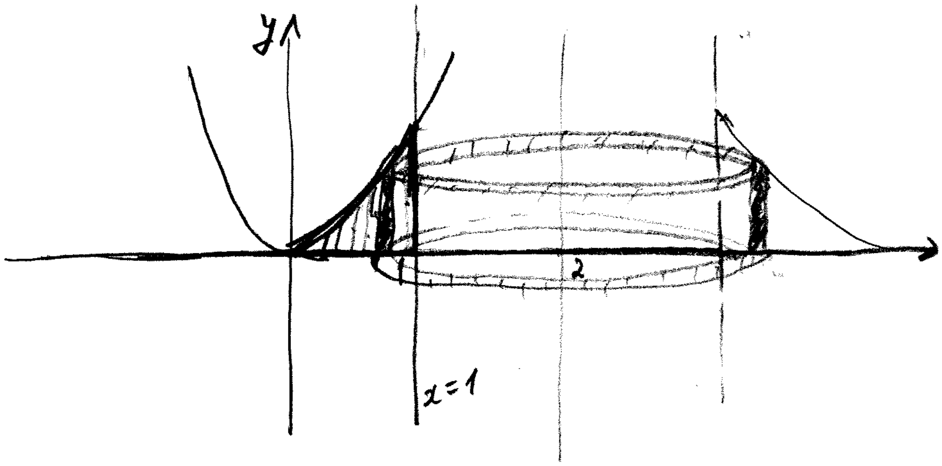
$$L = \int_0^3 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_0^3 \sqrt{1 + \frac{9}{4}x}\,dx = \frac{4}{9}\int_0^3 \left(1 + \frac{9}{4}x\right)^{1/2} d\left(1 + \frac{9}{4}x\right)$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2}\Big|_0^3 = \frac{8}{27}\left(1 + \frac{9}{4}\cdot 3\right)^{3/2} - \frac{8}{27}\cdot 1$$

$$= \frac{8}{27}\left(\frac{31}{4}\right)^{3/2} - \frac{8}{27} = \frac{8}{27}\left[\left(\frac{\sqrt{31}}{2}\right)^3 - 1\right].$$

Question 3. [10 pts] Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = x^4$, $y = 0$, $x = 1$ about the line $x = 2$.



$$V = \int_a^b 2\pi x f(x)\,dx$$

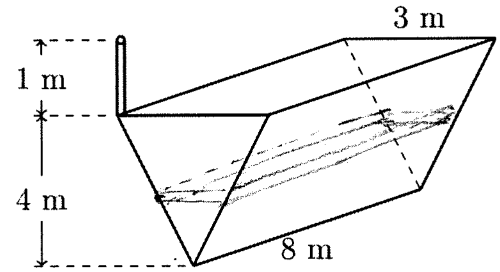
$$V = \int_0^1 2\pi(2-x)x^4\,dx = \int_0^1 2\pi(2x^4 - x^5)\,dx$$

$$= 2\pi\left[\frac{2}{5}x^5 - \frac{x^6}{6}\right]_0^1 = 2\pi\left[\frac{2}{5} - \frac{1}{6}\right]$$

$$= 2\pi \cdot \frac{7}{30} = \frac{7\pi}{15}$$

Question 4. [10 pts] A reservoir in the form of a straight prism with triangular base is shown in the figure to the right.

Its vertical faces are isosceles triangles of height 4 m and base 3 m, its length is 8 m, it is near the surface of the Earth, and it is full of water, which will be pumped to a height of 1 m above the reservoir.



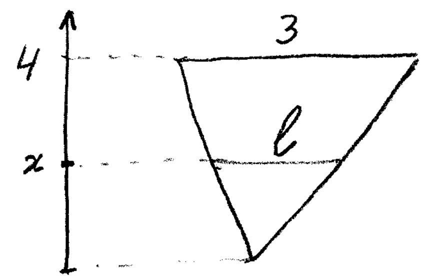
Denote by x the height in meters measured **from the bottom of the reservoir**.

a) What is, at first approximation, the volume of the layer of water between the heights x and $x + \Delta x$?

b) What is, at first approximation, the work required to pump that layer of water to a height of 1 m above the reservoir? Recall that the density of water is 1000 kg/m^3, and gravitational acceleration at the surface of the earth is $g \cong 9.8$ m/s^2.

c) What is, in Joules, the work required to pump all the water from the reservoir to a height of 1 m above the reservoir?

(a) A horizontal cross section at height x is a rectangle with length 8 and width ℓ. Using similar triangles, we find ℓ



$$\frac{\ell}{x} = \frac{3}{4} \Rightarrow \ell = \frac{3x}{4}$$

The volume of the layer $\Delta V \approx$ (area of the layer) $\times$ (Δx)

$$\Delta V = 8 \cdot \frac{3x}{4} \Delta x = 6x\Delta x$$

(b) Work $\approx$ Force $\times$ Distance $= 1000g(\Delta V) \times$ (distance)

$$\Delta W \approx 1000 \cdot (9.8)(6x\Delta x)(5-x) = 58800x(5-x)\Delta x$$

(c) $$W = \int_0^4 58800x(5-x)\,dx = 58800\int_0^4 (5x - x^2)\,dx$$

$$= 58800\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^4 = 58800\left(\frac{5 \cdot 16}{2} - \frac{64}{3}\right)$$

$$= 1,097,600$$

Question 5. [4 pts] Use Euler's method with step $h = 0.1$ to estimate $y(0.2)$, where $y(x)$ is the solution of the initial value problem $y' = y + xy$, $\quad y(0) = 1$.

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_0 = 1, \quad x_0 = 0$$

at $x_1 = 0.1$

$$y_1(0.1) = y_0 + (0.1) \cdot (y_0 + x_0 y_0)$$

$$= 1 + (0.1)(1 + 0.1) = 1.1$$

at $x = 0.2$

$$y_2(0.2) = y_1 + (0.1) \cdot (y_1 + x_1 \cdot y_1)$$

$$= 1.1 + (0.1)[1.1 + (0.1)(1.1)]$$

$$= 1.1 + (0.1)(1.1 + 0.11)$$

$$= 1.1 + (0.1)(1.21)$$

$$= 1.1 + 0.121$$

$$= 1.221$$

Question 6.

a) [3 pts] Solve the differential equation $xy^2y' = x + 1$. Write down the solution in explicit form.

b) [5 pts] Solve the initial value problem $\frac{dy}{dx} = 4(y^2 + 1)$, $y(\pi/4) = 1$.

(a) $xy^2 \frac{dy}{dx} = x + 1$

$$\int y^2 dy = \int \frac{x+1}{x} dx + C$$

$$\frac{y^3}{3} = x + \ln x + C$$

$$y = \left[3(x + \ln x + C)\right]^{1/3}$$

(b) $\frac{dy}{dx} = 4(y^2 + 1)$

$$\int \frac{dy}{y^2+1} = \int 4\,dx + C$$

$$\arctan y = 4x + C$$

$y = \tan(4x + C)$ General Solution

$$y\left(\frac{\pi}{4}\right) = \tan\left(4 \cdot \frac{\pi}{4} + C\right) = 1$$

$$\Rightarrow \pi + C = \frac{\pi}{4}$$

$$C = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

so, the solution is

$$\boxed{y = \tan\left(4x - \frac{3\pi}{4}\right)}$$