I Page 6 3, I put down that the 3 eqns = 0 but the took marks off, also, I put down an explanation that states its spanning set but didn't get marks, why?

DEPARTMENT OF MATHEMATICS

MIDTERM TEST #2

MTH 141 – LINEAR ALGEBRA

Last Name (Print[redacted] First Na[redacted] Student Numbe[redacted]

Signature: [redacted]

Date: Nov [redacted], 2013, 4:00 pm

Duration: 1.5 hours

Section (circle one)

Dr. Alvarez :	1	2	3	4	5
Dr. Liu :	6	7	8	9	10
Dr. Jung :	11	12	13	14	15
Dr. Liu:	16	17	18	19	20
Dr. Fisseha :	21	22	23	24	25

Instructions:

1. This is a closed-book test. **Notes, calculators and other aids are not permitted.**
2. Verify that your test has pages 1-7.
3. (a) Unless otherwise instructed, **make sure you include all significant steps in your solution, presented in the correct order. Unjustified answers will be given little or no credit. Cross out or erase all rough work not relevant to your solution.** Put a box around your final **answer.**

   (b) For multiple choice questions make sure to write your answers in the box at the end of each question **carefully**. There are no part marks in the multiple-choice section and **only** the answer in the box will be marked. The correct response gets full marks, an incorrect response or no response gets no marks.

   (c) Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there. Marks (out of 50) are shown in brackets.
4. Do not separate the sheets.
5. Have your student card available on your desk.

For Instructor's use only.

Page	Mark	
MC	4 /6	
3	8 /10	
4	8 /8	
5	5 /8	+2
6	6 /6	
7	8 /12	
Total	39 /50	+2

OK

$$\text{refl}_n = e - 2\left(\frac{e \cdot n}{\|n\|^2}\right)n$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rank + null = n

rank = dim

1. [2 marks] (Multiple choice question). Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping with standard matrix $[L] = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}$. The domain and codomain of L are respectively

A) $\mathbb{R}^2$ and $\mathbb{R}^3$ B) $\mathbb{R}^3$ and $\mathbb{R}^3$ C) $\mathbb{R}^3$ and $\mathbb{R}^2$ D) $\mathbb{R}^2$ and $\mathbb{R}^2$ E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. $\boxed{\text{C)}}$

$D = \mathbb{R}^3$

$CD = \mathbb{R}^2$

4 2. [2 marks] (Multiple choice question). The 4×5 matrix A has nullity equal to 2. The rank of A is equal to

A) 2 B) 3 C) 4 D) 5 E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. $\boxed{\text{B)}}$

1
2
3
5 4 3 2 1

4 rows
5 cols

rank = 5 − 2
= 3

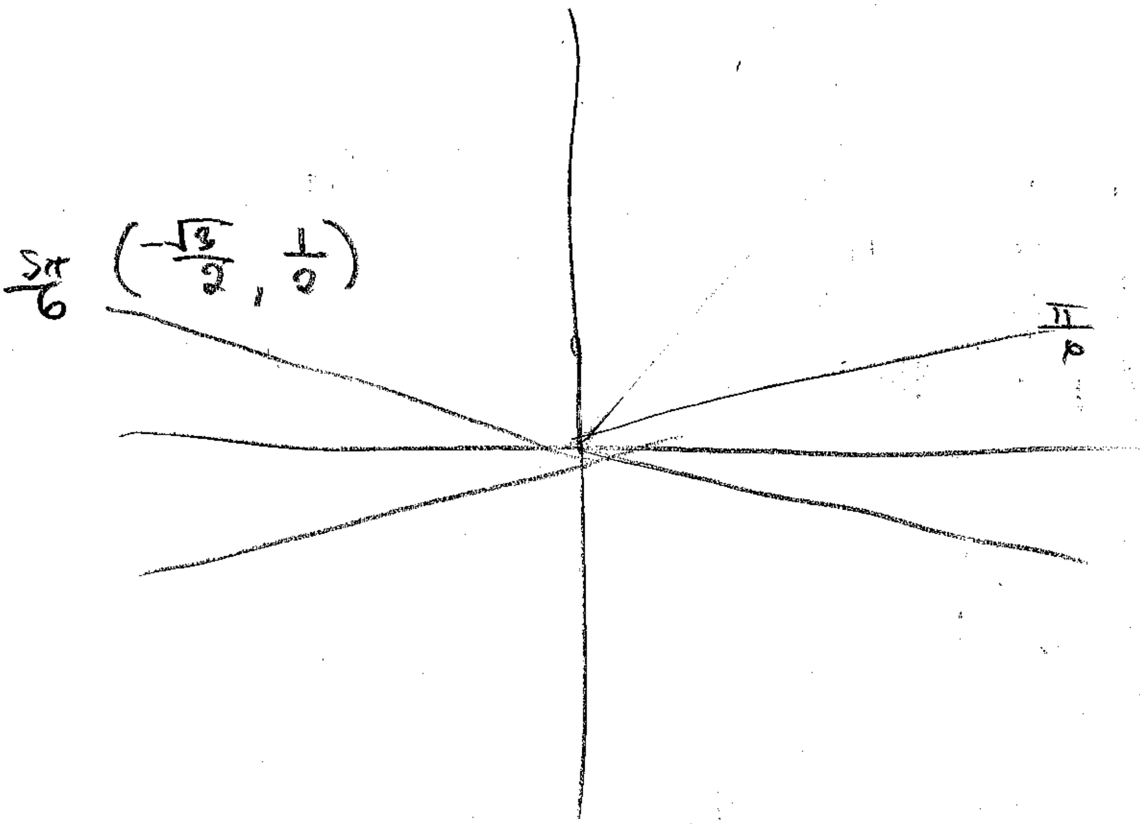
3. [2 marks] (Multiple choice question). Square matrices M and N satisfy that $M^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $N^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

A) $(MN)^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ B) $(MN)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ C) $(MN)^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

D) $(MN)^{-1}$ Does not exist E) None of these

Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. $\boxed{\text{B)}}$



b) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix}$

4. [10 marks] Let $R_{5\pi/6}$ be the rotation in the plane through the angle $\theta = 5\pi/6$

a) Determine the matrix of $R_{5\pi/6}$

b) Determine the matrix of the linear mapping that results from the composition of a stretch by a factor of 2 in the x_1 direction followed by a rotation through angle $\theta = 5\pi/6$

a) $[R_{\frac{5\pi}{6}}] = \begin{bmatrix} \cos\frac{5\pi}{6} & -\sin\frac{5\pi}{6} \\ \sin\frac{5\pi}{6} & \cos\frac{5\pi}{6} \end{bmatrix}$

$$\boxed{[R_{\frac{5\pi}{6}}] = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}$$

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b) $[S_{2x_1}][R_{\frac{5\pi}{6}}] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

$$= \begin{bmatrix} \left(\frac{-2\sqrt{3}}{2} + 0\left(\frac{1}{2}\right)\right) & \left(\frac{-2}{2} + 0\left(\frac{\sqrt{3}}{2}\right)\right) \\ \left(0\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\right) & \left(0\left(-\frac{1}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2\sqrt{3}}{2} & -\frac{2}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$$

4

$$\boxed{[S_{2x_1}][R_{\frac{5\pi}{6}}] = \begin{bmatrix} -\sqrt{3} & -1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}$$

5 b) If $M\vec{x} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$, then

$$\vec{x} = M^{-1} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ -1 & \frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} (1(0) + 0(3) - \frac{1}{2}(-2)) \\ (-1(0) + \frac{1}{2}(3) + \frac{3}{4}(-2)) \\ (1(0) - \frac{1}{2}(3) - \frac{1}{4}(-2) \end{bmatrix}$$

$$= \begin{bmatrix} (0 + 0 + 1) \\ (0 + \frac{3}{2} - \frac{3}{2}) \\ (0 - \frac{3}{2} + \frac{1}{2} \end{bmatrix}$$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

Check

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0 - 1 \\ 2 + 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

5. [8 marks] Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$

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a) Determine the inverse M^{-1} using row-reduction. No credit will be given if a different method is used to find the inverse.

b) Use the result in part a) to solve $M\vec{x} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. No credit will be given if a different method is used to solve the system.

a) $M^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] R_2 - 2R_1$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ -R_2 \\ R_3 + 2R_2 \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & -4 & -4 & 2 & 1 \end{array}\right] \frac{-R_3}{4}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{4} \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ R_2 - 3R_3 \\ \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{4} \end{array}\right]$

$\boxed{M^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ -1 & \frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}}$

b)

6. [8 marks] It is known that the vector space P_2, consisting of all polynomials of degree at most 2, has dimension 3. Consider the following set of polynomials in P_2: $\mathcal{B} = \{1 + x + 2x^2, 2 + x, 3 - x^2\}$

a) Prove that the set of polynomials $\mathcal{B}$ is linearly independent

b) Is $\mathcal{B}$ a spanning set for P_2?. Explain.

a) $\beta = \{1 + x + 2x^2, 2 + x, 3 - x^2\}$

$$P(x) = t_1(1 + x + 2x^2) + t_2(2 + x) + t_3(3 - x^2) \equiv 0$$
$$= t_1 + t_1 x + 2t_1 x^2 + 2t_2 + t_2 x + 3t_3 - t_3 x^2$$

$$t_1 + 2t_2 + 3t_3 = 0$$
$$t_1 x + t_2 x = 0$$
$$2t_1 x^2 - t_3 x^2 = 0$$

$\therefore$ Matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

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3 + 1

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & -4 & -7 \end{bmatrix} -R_2 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & -4 & -7 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 4R_2 \end{matrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix} \frac{R_3}{5}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 3R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix} \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\therefore$ the set of polynomials β is linearly independent

why?

0.5

b) β is a spanning set for P_2 since it adheres to the fact that it is linearly independent and is consistent. It also has 3 leading 1 coefficient means that its rank and dimension are both 3 / as stated in the original question. It also has a domain of 3 which makes sense b/c the set of polynomials in P_2 should only go up to value that are less than or equal to degree 2

$\therefore$ it is a spanning set as all polynomials can be represented using these polynomials

2

7. [6 marks] Consider the following basis of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Find the coordinates of the vector $x = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ with respect to the basis $\mathcal{B}$

$\therefore$ since $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ and $x = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ we should get the matrix: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & -1 & -1 \end{array}\right]$. Put into RREF to get $[\vec{x}]_\beta$

6

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & -1 & -1 \end{array}\right] R_3 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -4 \end{array}\right] \frac{-R_3}{2} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]}$$

$\therefore$ the coordinates of vector x with respect to basis β would be

$$\boxed{[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$$

8. [12 marks] Consider

$$S = \left\{ A \in M(2,2) : A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

a) Prove that S is a subspace of $M(2,2)$

b) Find a basis for S

c) What is the dimension of S? Explain.

8/12

a) We have the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for Matrix A, where $a,b,c,d \in \mathbb{R}$

Following the condition $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we could ~~get~~ the equation through multiplication:

$$\boxed{a - 2b = 0 \quad \text{and} \quad c - 2d = 0}$$

5/5

Now we must prove that it is closed under addition and scalar multiplication to be a subspace. Note that the condition clearly states that the multiplication of A and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ equals the zero vector $(\vec{0})$ $\therefore$ there is no need to prove that it contains the 0 vector since it states that it already does.

Closed Under Addition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then

$$A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}, \text{ then we get}$$

$(a+e) - 2(b+f) = 0$ and $(c+g) - 2(d+h) = 0$

$a + e - 2b - 2f = 0$ $\qquad$ $c + g - 2d - 2h = 0$

$(a-2b) + (e-2f) = 0$ $\qquad$ $(c-2d) + (g-2h) = 0$

$0 + 0 = 0$ $\qquad$ $0 + 0 = 0$

$0 = 0$ $\qquad$ $0 = 0$

Closed Under Scalar Multiplication

Let $t \in \mathbb{R}$, then

$t(a-2b) = ta - 2tb$ and $t(c-2d) = tc - 2td$

$t(0) = t(a-2b)$ $\qquad$ $t(0) = t(c-2d)$

$0 = t(0)$ $\qquad$ $0 = t(0)$

$0 = 0$ $\qquad$ $0 = 0$

$\therefore$ Since it contains the $\vec{0}$, closed under scalar multiplication & addition, S is a subspace of $M(2,2)$

b) & c) on back