MAT 2384 Assignment # 5 Solutions

1. $y'' + y = 2\sec x \quad , \quad y(0) = y'(0) = 2$

the corresponding homogeneous DE is $y'' + y = 0$

and so $y_h(x) = C_1 \cos x + C_2 \sin x$

(ie $y_1(x) = \cos x$ and $y_2(x) = \sin x$)

$r(x) = 2\sec x \Rightarrow$ must use Variation of Parameters

we have to solve: $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = r(x)$

or $u_1' \cos x + u_2' \sin x = 0$ ①

$-u_1' \sin x + u_2' \cos x = 2\sec x$ ②

① $\times \sin x \Rightarrow u_1' \sin x \cos x + u_2' \sin^2 x = 0$ ③

② $\times \cos x \Rightarrow -u_1' \sin x \cos x + u_2' \cos^2 x = 2$ ④

so ③ + ④ $\Rightarrow u_2' = 2 \Rightarrow u_2(x) = 2x$

then ① $\Rightarrow u_1' = -u_2' \dfrac{\sin x}{\cos x} = -2\dfrac{\sin x}{\cos x} \Rightarrow u_1(x) = 2\ln|\cos x|$

and so the particular solution is

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$
$$= 2\cos x \ln|\cos x| + 2x \sin x$$

and the general solution is

$$y_g(x) = y_h(x) + y_p(x) = C_1 \cos x + C_2 \sin x + 2\cos x \ln|\cos x| + 2x\sin x$$

$y(0) = 2 \Rightarrow 2 = C_1 \cos(0) + C_2 \sin(0) + 2\cos(0)\ln|\cos(0)| + 2(0)\sin(0)$

and so $C_1 = 2$

$$y_g'(x) = -C_1 \sin x + C_2 \cos x - 2\sin x \ln|\cos x| - 2\sin x + 2\sin x + 2x\cos x$$

$y'(0) = 2 \Rightarrow 2 = -C_1 \sin(0) + C_2 \cos(0) - 2\sin(0)\ln|\cos(0)| + 2(0)\cos(0)$

which means $C_2 = 2$ and the unique solution is

$$\boxed{y(x) = 2\cos x + 2\sin x + 2\cos x \ln|\cos x| + 2x\sin x}$$

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2. $x^2y'' - 2xy' + 2y = x^2$, $x>0$. $y(1)=3$, $y'(1)=5$

corresp. homog. DE is $x^2y'' - 2xy' + 2y = 0$

which has char. eq. $m(m-1) - 2m+2 = m(m-1) - 2(m-1)$

$= (m-1)(m-2) = 0 \Rightarrow m_1 = 1,\ m_2 = 2$

and so $y_h(x) = c_1 x + c_2 x^2$ ($\Rightarrow y_1(x) = x,\ y_2(x) = x^2$)

EC DE $\Rightarrow$ use Var of Par's and $r(x) = 1$ (std form!)

so we solve $u_1' x + u_2' x^2 = 0$ or $u_1' + u_2' x = 0$ ①

$u_1' + 2u_2' x = 1$ $\qquad u_1' + 2u_2' x = 1$ ②

② − ① $\Rightarrow u_2' x = 1 \Rightarrow u_2' = 1/x \Rightarrow u_2(x) = \ln|x|$

then ① $\Rightarrow u_1' = -u_2' x = -1 \Rightarrow u_1(x) = -x$

then $y_p(x) = (-x)(x) + (x^2)(\ln|x|) = -x^2 + x^2 \ln|x|$

but x^2 appears in $y_h(x)$, so we can take $y_p(x) = x^2 \ln|x|$

so the general solution is $y_g(x) = c_1 x + c_2 x^2 + x^2 \ln|x|$

then $y(1)=3 \Rightarrow 3 = c_1(1) + c_2(1)^2 + (1)^2 \ln(1) \Rightarrow c_1 + c_2 = 3$

$y_g'(x) = c_1 + 2c_2 x + 2x \ln|x| + x$

so $y'(1) = 5 \Rightarrow 5 = c_1 + 2c_2(1) + 2(1)\ln(1) + 1 \Rightarrow c_1 + 2c_2 = 4$

then $c_1 = 2$ and $c_2 = 1$ and the unique solution

is $\boxed{y(x) = 2x + x^2 + x^2 \ln|x|}$

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3. $y'' - 4y' + 4y = 3x^{-2}e^{2x}$, $y(1) = y'(1) = 2e^2$

corresp. homog. DE is $y'' - 4y' + 4y = 0$, with char. eq.

$\lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$ and so

$y_h(x) = c_1 e^{2x} + c_2 x e^{2x}$ (ie $y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$)

$r(x) = 3x^{-2}e^{2x} \Rightarrow$ need Var of Par's

so we solve $u_1' e^{2x} + u_2' x e^{2x} = 0$

$2u_1' e^{2x} + u_2' (e^{2x} + 2xe^{2x}) = 3x^{-2} e^{2x}$

or $\quad u_1' + u_2' x = 0 \quad ①$

$\quad 2u_1' + u_2'(1+2x) = 3x^{-2} \quad ②$

then $② - 2\times① \Rightarrow \quad u_2' = 3x^{-2} \Rightarrow \quad u_2(x) = -3x^{-1}$

then $① \Rightarrow \quad u_1' = -u_2' x = -3x^{-1} \quad \Rightarrow \quad u_1(x) = -3\ln|x|$

and thus $y_p(x) = (-3\ln|x|)(e^{2x}) + (-3x^{-1})(xe^{2x})$

$\qquad = -3e^{2x}\ln|x| - 3e^{2x}$

but e^{2x} appears in y_h, so we take $\quad y_p(x) = -3e^{2x}\ln|x|$

and the general solution is $y_g(x) = c_1 e^{2x} + c_2 x e^{2x} - 3e^{2x}\ln|x|$

$y(1) = 2e^2 \Rightarrow \quad 2e^2 = c_1 e^2 + c_2(1)e^2 - 3e^2\ln(1) \quad \Rightarrow \quad c_1 + c_2 = 2$

$y_g'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x} - 6e^{2x}\ln|x| - 3e^{2x}/x$

so $y'(1) = 2e^2 \Rightarrow \quad 2e^2 = 2c_1 e^2 + c_2 e^2 + 2c_2(1)e^2 - 6e^2\ln(1) - 3e^2/(1)$

and so $\quad 2c_1 + 3c_2 = 5$

so then $\quad c_1 = c_2 = 1 \quad$ and the unique solution is

$$\boxed{y(x) = e^{2x} + xe^{2x} - 3e^{2x}\ln|x|}$$

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4. $\quad x^3 y''' - 6xy' + 12y = 20x^4, \quad x > 0, \quad y(1) = 8/3, \quad y'(1) = 56/3, \quad y''(1) = 14$

correspd. homog. DE is $\quad x^3 y''' - 6xy' + 12y = 0$, which has

char. eq. $\quad m(m-1)(m-2) - 6m + 12 = m(m-1)(m-2) - 6(m-2)$

$\quad = (m-2)[m(m-1) - 6] = (m-2)(m^2 - m - 6)$

$\quad = (m-2)(m+2)(m-3) = 0 \quad \Rightarrow \quad m_{1,2} = \pm 2, \; m_3 = 3$

and so $\quad y_h(x) = c_1 x^2 + c_2 x^{-2} + c_3 x^3 \quad (y_1(x) = x^2,\; y_2(x) = x^{-2},\; y_3(x) = x^3)$

ECDE $\Rightarrow$ use Var of Par's and $\quad r(x) = 20x \quad$ (std form)

so we solve $\quad u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$

$\quad u_1' y_1' + u_2' y_2' + u_3' y_3' = 0$

$\quad u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = r$

so $u_1' x^2 + u_2' x^{-2} + u_3' x^3 = 0$ (1)

$2u_1' x - 2u_2' x^{-3} + 3u_3' x^2 = 0$ (2)

$2u_1' + 6u_2' x^{-4} + 6u_3' x = 20x$ (3)

(3) $\times x \Rightarrow 2u_1' x + 6u_2' x^{-3} + 6u_3' x^2 = 20x^2$ (4)

(4) $-$ (2) $\Rightarrow 8u_2' x^{-3} + 3u_3' x^2 = 20x^2$ (5)

(2) $\times x \Rightarrow 2u_1' x^2 - 2u_2' x^{-2} + 3u_3' x^3 = 0$ (6)

(6) $- 2\times$ (1) $\Rightarrow -4u_2' x^{-2} + u_3' x^3 = 0$ (7)

(5) $\times x \Rightarrow 8u_2' x^{-2} + 3u_3' x^3 = 20x^3$ (8)

(8) $+ 2\times$ (7) $\Rightarrow 5u_3' x^3 = 20x^3 \Rightarrow u_3' = 4 \Rightarrow u_3(x) = 4x$

then (7) $\Rightarrow u_2' x^{-2} = \frac{1}{4} u_3' x^3 = x^3 \Rightarrow u_2' = x^5 \Rightarrow u_2(x) = \frac{1}{6} x^6$

and (1) $\Rightarrow u_1' x^2 = -u_2' x^{-2} - u_3' x^3 = -x^3 - 4x^3 = -5x^3$

so $u_1' = -5x \Rightarrow u_1(x) = -\frac{5}{2} x^2$

so $y_p(x) = (-\frac{5}{2}x^2)(x^2) + (\frac{1}{6}x^6)(x^{-2}) + (4x)(x^3) = \frac{5}{3} x^4$

and the general solution is $y_g(x) = c_1 x^2 + c_2 x^{-2} + c_3 x^3 + \frac{5}{3} x^4$

$y(1) = 8/3 \Rightarrow 8/3 = c_1(1)^2 + c_2(1)^{-2} + c_3(1)^3 + \frac{5}{3}(1)^4 \Rightarrow c_1 + c_2 + c_3 = 1$ (A)

$y_g'(x) = 2c_1 x - 2c_2 x^{-3} + 3c_3 x^2 + \frac{20}{3} x^3$

$y'(1) = 50/3 \Rightarrow 50/3 = 2c_1(1) - 2c_2(1)^{-3} + 3c_3(1)^2 + \frac{20}{3}(1)^3 \Rightarrow 2c_1 - 2c_2 + 3c_3 = 10$ (B)

$y_g''(x) = 2c_1 + 6c_2 x^{-4} + 6c_3 x + 20x^2$

$y''(1) = 14 \Rightarrow 14 = 2c_1 + 6c_2(1)^{-4} + 6c_3(1) + 20(1)^2 \Rightarrow 2c_1 + 6c_2 + 6c_3 = -6$ (C)

(C) $- 6\times$ (A) $\Rightarrow -4c_1 = -12 \Rightarrow c_1 = 3$

so (A) becomes $c_2 + c_3 = -2$ (D) $2\times$(D)$+$(E) $\Rightarrow c_3 = 0 \Rightarrow c_2 = -2$

(B) becomes $-2c_2 + 3c_3 = 4$ (E)

$\therefore$ the unique solution is

$$\boxed{y(x) = 3x^2 - 2x^{-2} + \frac{5}{3} x^4}$$

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5. $\int_0^4 \frac{2x}{1+x^2}\,dx$ with Simpson with $2n=8$

so $h = \frac{4-0}{8} = 1/2$, $x_0=0$, $x_1=0.5$, $x_2=1$, $x_3=1.5$, $x_4=2$, $x_5=2.5$, $x_6=3$, $x_7=3.5$ and $x_8=4$

$$\int_0^4 \frac{2x}{1+x^2}\,dx \approx \frac{h}{3}\sum_{j=0}^{3}\left(f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})\right)$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left[\frac{2(0)}{1+(0)^2} + 4\left(\frac{2(0.5)}{1+(0.5)^2}\right) + 2\left(\frac{2(1)}{1+(1^2)}\right) + 4\left(\frac{2(1.5)}{1+(1.5)^2}\right) + 2\left(\frac{2(2)}{1+(2)^2}\right) + 4\left(\frac{2(2.5)}{1+(2.5)^2}\right) + 2\left(\frac{2(3)}{1+(3)^2}\right) + 4\left(\frac{2(3.5)}{1+(3.5)^2}\right) + \frac{2(4)}{1+(4)^2}\right]$$

$$= \frac{1}{6}\left[0 + 3.2 + 2 + 3.6923077 + 1.6 + 2.7586207 + 1.2 + 2.1132075 + 0.4705882\right]$$

$$= \boxed{2.839121}$$

true value: $\int_0^4 \frac{2x}{1+x^2}\,dx = \ln(1+x^2)\Big|_0^4 = \ln(17) = 2.833213$

so the error is $\epsilon = |2.839121 - 2.833213| = \boxed{0.005908}$

(which is 0.21% - ie the result is good)