



Université d'Ottawa · University of Ottawa

Faculté des sciences / Faculty of Science
Mathématiques et de statistique / Mathematics and Statistics

Calculus III for Engineers

MAT 2322A - Fall 2015

Midterm II

Professor: Victor G. LeBlanc

Time limit: 80 minutes. Closed books.

Name: Solution ID Number: _____

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

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Instructions

- The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.
- The exam has 8 pages. Read each question carefully before answering.
- Questions 1 to 3 are multiple choice. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 4 to 6 are long answer questions. Questions 4 and 6 are worth 6 marks each, and question 5 is worth 7 marks, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test. Good luck!

Answers to multiple choice Qs

1	2	3
F	A	B

Grid below is used for grading
(do not write in this grid)

MCQ	4	5	6	Total
	/6	/6	/7	/6
				/25

1. Consider the parametric surface

$$\vec{r}(p, q) = (\cos q)\vec{i} + p\vec{j} + (\sin q)\vec{k}, \quad p \in [2, 5], q \in [0, \pi/2].$$

What is the total surface area of this parametric surface?

A. π

B. $\frac{5\pi}{3}$

C. $\frac{5\pi}{2}$

D. 2π

E. $\frac{\pi}{2}$

F. $\frac{3\pi}{2}$

$$\vec{r}_p = \vec{j}, \quad \vec{r}_q = -\sin q \vec{i} + \cos q \vec{j}$$

$$\vec{r}_p \times \vec{r}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\sin q & 0 & \cos q \end{vmatrix} = \vec{i} \cos q + \vec{k} \sin q$$

$$\|\vec{r}_p \times \vec{r}_q\| = \sqrt{\cos^2 q + \sin^2 q} = \sqrt{1} = 1$$

$$\text{Area} = \int_2^5 \int_0^{\pi/2} \|\vec{r}_p \times \vec{r}_q\| dq dp = \int_2^5 \int_0^{\pi/2} 1 dq dp = 3\pi/2$$

2. Find the total arc-length of the parametrized curve

$$\vec{r}(t) = \frac{1}{2}t^2\vec{i} + \frac{2\sqrt{2}}{3}t^{3/2}\vec{j} + t\vec{k}, \quad t \in [2, 3]$$

A. $\frac{7}{2}$

B. $\frac{3}{2}$

C. $\sqrt{2}$

D. π

E. $\frac{5}{2}$

F. 2

$$\vec{r}'(t) = t\vec{i} + \sqrt{2}t^{1/2}\vec{j} + \vec{k}$$

$$\|\vec{r}'(t)\| = \sqrt{t^2 + (\sqrt{2}t^{1/2})^2 + 1} = \sqrt{t^2 + 2t + 1} = \sqrt{(t+1)^2} = t+1$$

$$\text{Length} = \int_2^3 \|\vec{r}'(t)\| dt = \int_2^3 (t+1) dt = \frac{(t+1)^2}{2} \Big|_2^3$$

$$= \frac{4^2}{2} - \frac{3^2}{2} = \frac{7}{2}$$

3. A solid rectangular brick is bounded by the planes $z = 0$, $z = 3$, $x = 0$, $x = 2$, $y = 0$ and $y = 1$, and has mass density given by the function $\delta(x, y, z) = (x + 2y + z)$. What is the total mass of the brick?

A. 20

B. 21

C. 22

D. 23

E. 24

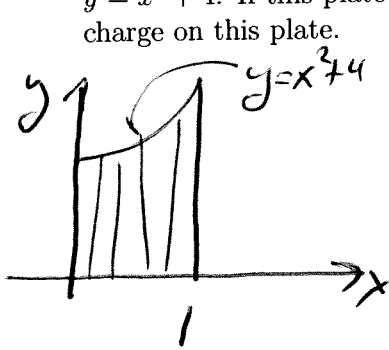
F. 25

$$\text{Mass} = \int_0^3 \int_0^2 \int_0^1 (x + 2y + z) \, dy \, dx \, dz = \int_0^3 \int_0^2 (xy + y^2 + yz)' \, dx \, dz$$

$$= \int_0^3 \int_0^2 (x + 1 + z) \, dx \, dz = \int_0^3 \left(\frac{x^2}{2} + x + xz \right)' \, dz$$

$$= \int_0^3 (2 + 2 + 2z) \, dz = \left(4z + z^2 \right)' \Big|_0^3 = 12 + 9 = 21$$

4.a) A 2-dimensional ceramic plate is bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = x^2 + 4$. If this plate has a charge density given by $\delta(x, y) = x^2 + 4y$, compute the total charge on this plate.



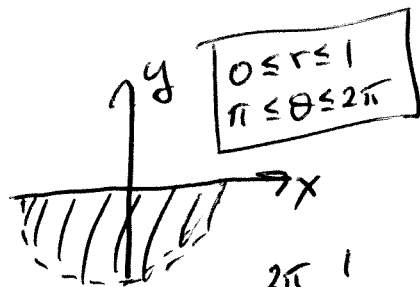
$$\text{Charge} = \int_0^1 \int_0^{x^2+4} (x^2+4y) dy dx = \int_0^1 \left[x^2 y + 2y^2 \right]_0^{x^2+4} dx$$

$$= \int_0^1 (x^2(x^2+4) + 2(x^2+4)^2) dx = \int_0^1 (x^4 + 4x^2 + 2x^4 + 16x^2 + 32) dx$$

$$= \int_0^1 (3x^4 + 20x^2 + 32) dx = \left[\frac{3x^5}{5} + \frac{20x^3}{3} + 32x \right]_0^1 = \frac{3}{5} + \frac{20}{3} + 32$$

$$= \frac{9}{15} + \frac{100}{15} + \frac{480}{15} = \frac{589}{15} = 39 \frac{4}{15}$$

4.b) Use polar coordinates to evaluate the double integral



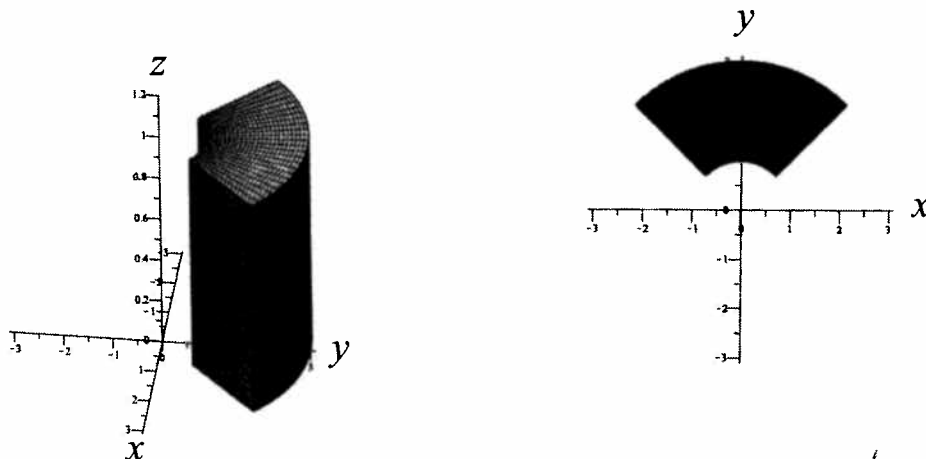
$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 y(x^2+y^2)^{3/2} dy dx = \int_{\theta=\pi}^{2\pi} \int_{r=0}^1 r \sin \theta (r^2)^{3/2} \cdot r dr d\theta$$

$$= \int_{\pi}^{2\pi} \int_0^1 r^5 \sin \theta dr d\theta = \int_{\pi}^{2\pi} \left[\frac{r^6}{6} \sin \theta \right]_{r=0}^1 d\theta = \int_{\pi}^{2\pi} \frac{1}{6} \sin \theta d\theta$$

$$= -\frac{1}{6} \cos \theta \Big|_{\pi}^{2\pi} = -\frac{1}{6} (\cos 2\pi - \cos \pi)$$

$$= -\frac{1}{6} (1 - (-1)) = -\frac{2}{6} = -\frac{1}{3}$$

5.a) The solid illustrated below (side view and top view) is bounded by the cylinder $x^2 + y^2 = 1$, the cylinder $x^2 + y^2 = 9$, the planes $z = 0$ and $z = 1$, and the surface $y = |x|$. If this solid has mass density given by $\delta(x, y, z) = (x^2 + y^2)yz$, compute the total mass by setting up and evaluating a triple integral in cylindrical coordinates.



In cylindrical coordinates, the region is described as
 $0 \leq z \leq 1$, $1 \leq r \leq 3$, $\pi/4 \leq \theta \leq 3\pi/4$.

$$\text{Mass} = \int_0^1 \int_{\pi/4}^{3\pi/4} \int_1^3 r^2 \cdot r \sin \theta \cdot z \cdot r \, dr \, d\theta \, dz = \int_0^1 \int_{\pi/4}^{3\pi/4} \int_1^3 r^4 z \sin \theta \, dr \, d\theta \, dz$$

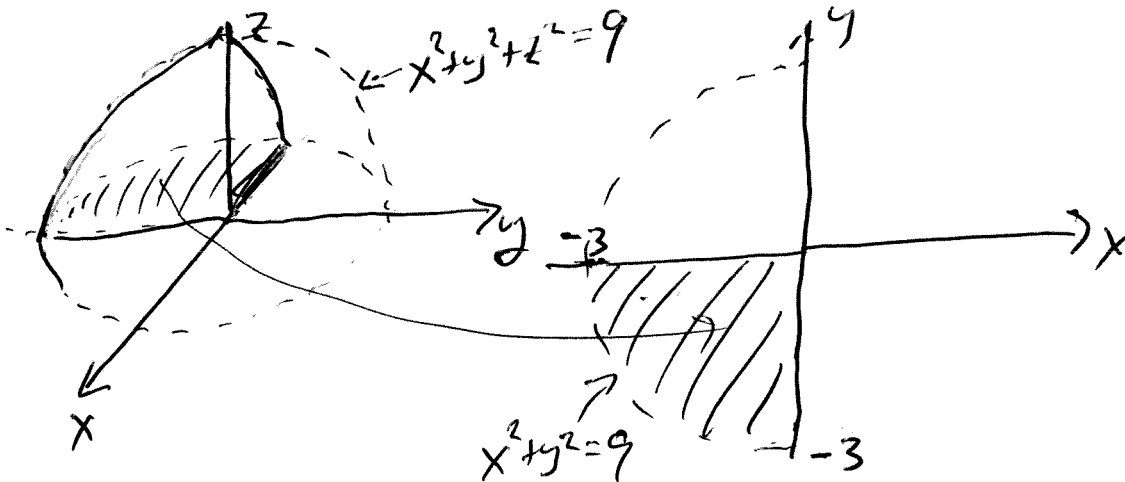
$$= \int_0^1 \int_{\pi/4}^{3\pi/4} \frac{r^5 z \sin \theta}{5} \Big|_1^3 \, d\theta \, dz = \int_0^1 \int_{\pi/4}^{3\pi/4} \frac{242 z \sin \theta}{5} \, d\theta \, dz = \int_0^1 \frac{-242 z \cos \theta}{5} \Big|_{\pi/4}^{3\pi/4} \, dz$$

$$= \int_0^1 -\frac{242 z}{5} \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \, dz = \int_0^1 \frac{242 \sqrt{2} z}{5} \, dz = \frac{242 \sqrt{2} z^2}{10} \Big|_0^1$$

$$= \frac{121\sqrt{2}}{5}$$

5.b) Convert the following triple integral into an equivalent triple integral using spherical coordinates, **BUT DO NOT EVALUATE THESE INTEGRALS**:

$$\int_{-3}^0 \int_{-\sqrt{9-y^2}}^0 \int_0^{\sqrt{9-x^2-y^2}} x(x^2+y^2+z^2)^{5/2} dz dx dy$$



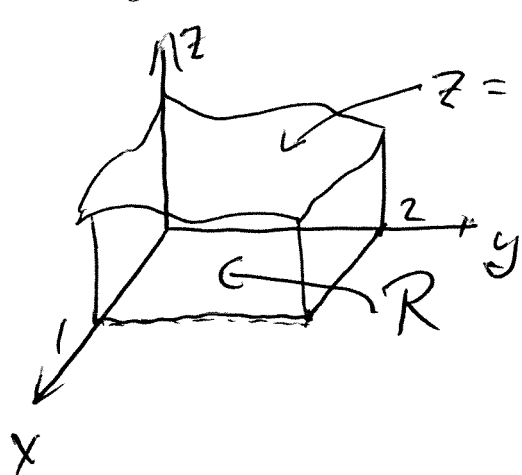
In spherical coordinates, this region is described as
 $0 \leq \rho \leq 3$, $0 \leq \phi \leq \pi/2$, $\pi \leq \theta \leq 3\pi/2$. So the

above integral becomes

$$\int_{\pi}^{3\pi/2} \int_0^{\pi/2} \int_0^3 \underbrace{\rho \sin \phi \cos \theta}_{x} (\rho^2)^{5/2} \underbrace{\rho^2 \sin \phi}_{dV \text{ in spherical}} d\rho d\phi d\theta$$

$$= \int_{\pi}^{3\pi/2} \int_0^{\pi/2} \int_0^3 \rho^8 \sin^2 \phi \cos \theta d\rho d\phi d\theta$$

6. Consider the 3-dimensional solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 2$ and the surface $z = \sqrt{50 - x^2 - 2y^2}$. If this solid has a mass density given by $\delta(x, y, z) = e^z(x + y)$, set-up a triple integral in cartesian coordinates (i.e. x, y, z) which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**



$$M_{\text{mass}} = \iiint_R \delta \, dV = \int_{y=0}^{y=2} \int_{x=0}^{x=1} \int_{z=0}^{z=\sqrt{50-x^2-2y^2}} e^z(x+y) \, dz \, dx \, dy$$

Note that on the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$, the function $50 - x^2 - 2y^2 > 0$.



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E	C	B

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MCQ	4	5	6	Total
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				/25

1. A solid rectangular brick is bounded by the planes $z = 0$, $z = 3$, $x = 0$, $x = 2$, $y = 0$ and $y = 1$, and has mass density given by the function $\delta(x, y, z) = (2x + y + z)$. What is the total mass of the brick?

A. 20

B. 21

C. 22

D. 23

E. 24

F. 25

$$\begin{aligned} \text{Mass} &= \int_0^3 \int_0^2 \int_0^1 (2x + y + z) \, dy \, dx \, dz = \int_0^3 \int_0^2 \left. 2xy + \frac{y^2}{2} + yz \right|_0^1 \, dx \, dz \\ &= \int_0^3 \int_0^2 \left(2x + \frac{1}{2} + z \right) \, dx \, dz = \int_0^3 \left. x^2 + \frac{1}{2}x + xz \right|_0^2 \, dz = \int_0^3 (4 + 1 + 2z) \, dz \\ &= 5z + z^2 \Big|_0^3 = 15 + 9 = 24 \end{aligned}$$

2. Consider the parametric surface

$$\vec{r}(p, q) = (\sin q)\vec{i} + p\vec{j} + (\cos q)\vec{k}, \quad p \in [1, 3], \quad q \in [0, \pi/2].$$

What is the total surface area of this parametric surface?

A. $\frac{\pi}{2}$

B. $\frac{3\pi}{2}$

C. π

D. $\frac{5\pi}{3}$

E. $\frac{5\pi}{2}$

F. 2π

This is $\frac{1}{4}$ of a cylindrical surface of height 2 and radius 1. So

$$\text{Area} = \frac{\pi}{2} \cdot 2 \cdot 1 = \pi$$

3. Find the total arc-length of the parametrized curve

$$\vec{r}(t) = \frac{1}{2}t^2\vec{i} + \frac{2\sqrt{2}}{3}t^{3/2}\vec{j} + t\vec{k}, \quad t \in [1, 2]$$

A. $\frac{3}{2}$

B. $\frac{5}{2}$

C. 2

D. $\frac{1}{2}$

E. $\sqrt{2}$

F. π

$$\vec{r}'(t) = t\vec{i} + \sqrt{2}t^{1/2}\vec{j} + \vec{k}$$

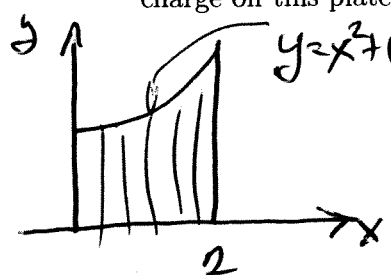
$$\|\vec{r}'(t)\| = \sqrt{t^2 + (\sqrt{2}t^{1/2})^2 + 1^2}$$

$$= \sqrt{t^2 + 2t + 1} = \sqrt{(t+1)^2} = t+1$$

$$\text{Length} = \int_1^2 \|\vec{r}'(t)\| dt = \int_1^2 (t+1) dt = \frac{(t+1)^2}{2} \Big|_1^2$$

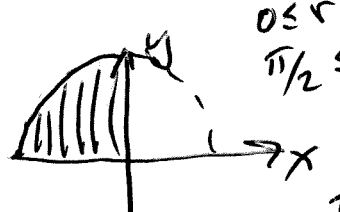
$$= \frac{3^2}{2} - \frac{2^2}{2} = \frac{5}{2}$$

4.a) A 2-dimensional ceramic plate is bounded by the curves $x = 0$, $x = 2$, $y = 0$ and $y = x^2 + 1$. If this plate has a charge density given by $\delta(x, y) = x^2 + 2y$, compute the total charge on this plate.



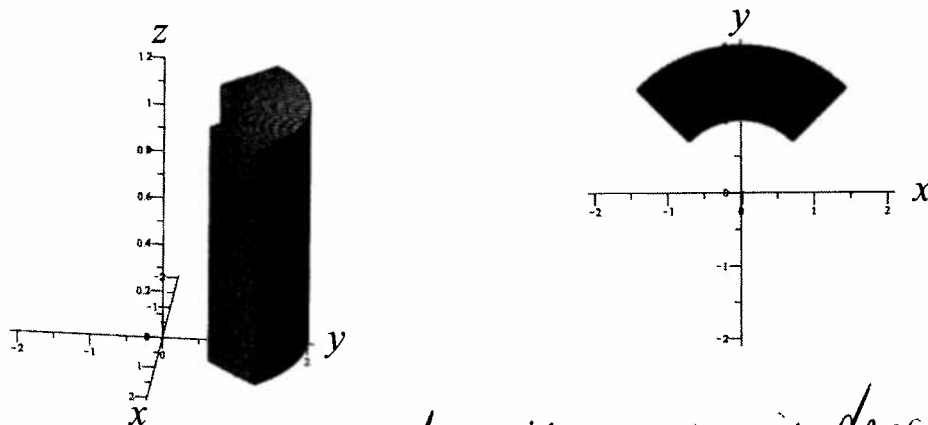
$$\begin{aligned} \text{Charge} &= \int_0^2 \int_0^{x^2+1} (x^2 + 2y) dy dx = \int_0^2 \left. x^2 y + y^2 \right|_0^{x^2+1} dx \\ &= \int_0^2 (x^2(x^2+1) + (x^2+1)^2) dx = \int_0^2 (x^4 + x^2 + x^4 + 2x^2 + 1) dx \\ &= \int_0^2 (2x^4 + 3x^2 + 1) dx = \left. \frac{2x^5}{5} + x^3 + x \right|_0^2 = \frac{64}{5} + 8 + 2 \\ &= \frac{64+50}{5} = \frac{114}{5} = 22\frac{4}{5} \end{aligned}$$

4.b) Use polar coordinates to evaluate the double integral



$$\begin{aligned} \int_{-1}^0 \int_0^{\sqrt{1-x^2}} x(x^2 + y^2)^{3/2} dy dx &= \int_{\theta=\pi/2}^{\pi} \int_{r=0}^1 r \cos \theta (r^2)^{3/2} \cdot r dr d\theta \\ &= \int_{\pi/2}^{\pi} \int_0^1 r^5 \cos \theta dr d\theta = \int_{\pi/2}^{\pi} \left. \frac{r^6}{6} \cos \theta \right|_0^1 d\theta = \int_{\pi/2}^{\pi} \frac{1}{6} \cos \theta d\theta \\ &= \left. \frac{1}{6} \sin \theta \right|_{\pi/2}^{\pi} = \frac{1}{6} (\sin \pi - \sin \pi/2) \\ &= \frac{1}{6} (-1) = -\frac{1}{6} \end{aligned}$$

5.a) The solid illustrated below (side view and top view) is bounded by the cylinder $x^2 + y^2 = 1$, the cylinder $x^2 + y^2 = 4$, the planes $z = 0$ and $z = 1$, and the surface $y = |x|$. If this solid has mass density given by $\delta(x, y, z) = (x^2 + y^2)yz$, compute the total mass by setting up and evaluating a triple integral in cylindrical coordinates.



In cylindrical coordinates, the region is described as
 $0 \leq z \leq 1$, $1 \leq r \leq 2$, $\pi/4 \leq \theta \leq 3\pi/4$

$$\text{Mass} = \int_0^1 \int_{\pi/4}^{3\pi/4} \int_1^2 r^2 \cdot r \sin \theta \cdot z \cdot r \, dr \, d\theta \, dz = \int_0^1 \int_{\pi/4}^{3\pi/4} \int_1^2 r^4 z \sin \theta \, dr \, d\theta \, dz$$

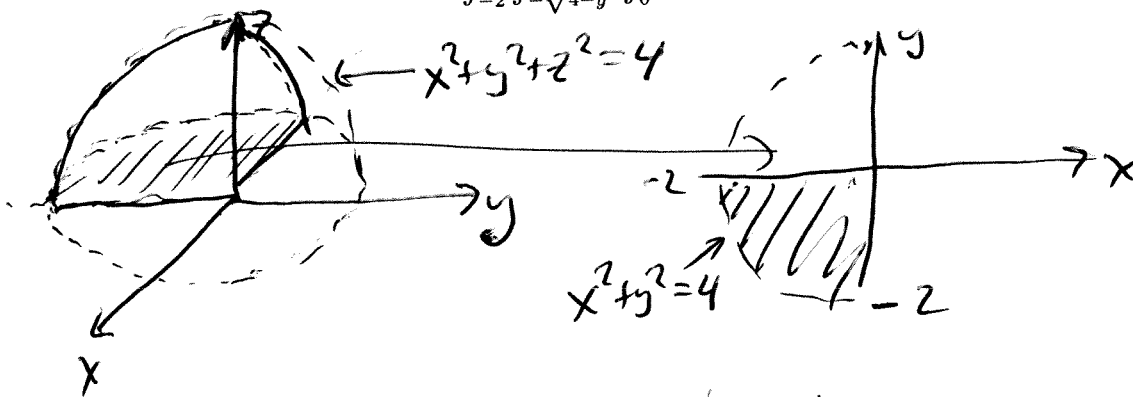
$$= \int_0^1 \int_{\pi/4}^{3\pi/4} \frac{r^5}{5} z \sin \theta \Big|_1^2 \, d\theta \, dz = \int_0^1 \int_{\pi/4}^{3\pi/4} \frac{31}{5} z \sin \theta \, d\theta \, dz = \int_0^1 \frac{-31}{5} z \cos \theta \Big|_{\pi/4}^{3\pi/4} \, dz$$

$$= \int_0^1 -\frac{31}{5} z \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \, dz = \int_0^1 \frac{31\sqrt{2}}{5} z \, dz = \frac{31\sqrt{2}}{10} z^2 \Big|_0^1$$

$$= \frac{31\sqrt{2}}{10}$$

5.b) Convert the following triple integral into an equivalent triple integral using spherical coordinates, **BUT DO NOT EVALUATE THESE INTEGRALS:**

$$\int_{-2}^0 \int_{-\sqrt{4-y^2}}^0 \int_0^{\sqrt{4-x^2-y^2}} y(x^2 + y^2 + z^2)^{3/2} dz dx dy$$

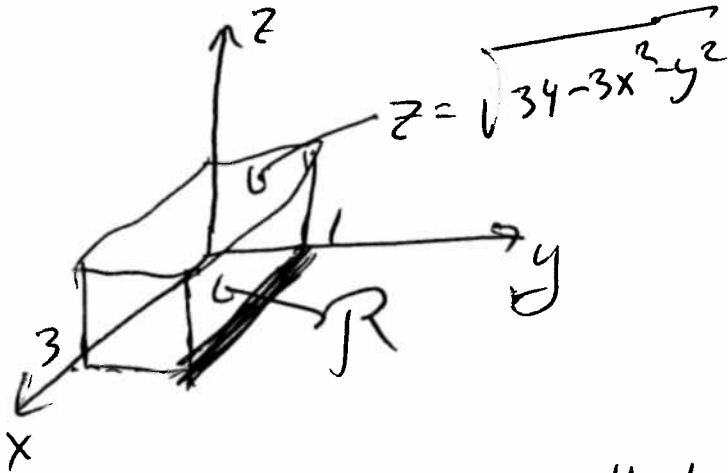


In spherical coordinates, this region is described as $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/2$, $\pi \leq \theta \leq 3\pi/2$. So the above integral becomes

$$\int_{\pi}^{3\pi/2} \int_0^{\pi/2} \int_0^2 \underbrace{\rho \sin \phi \sin \theta}_{y} (\rho^2)^{3/2} \cdot \underbrace{\rho^2 \sin \phi d\rho d\phi d\theta}_{dV \text{ in spherical}}$$

$$= \int_{\pi}^{3\pi/2} \int_0^{\pi/2} \int_0^2 \rho^6 \sin^2 \phi \sin \theta d\rho d\phi d\theta$$

6. Consider the 3-dimensional solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 3$, $y = 1$ and the surface $z = \sqrt{34 - 3x^2 - y^2}$. If this solid has a mass density given by $\delta(x, y, z) = e^z(2x + y)$, set-up a triple integral in cartesian coordinates (i.e. x, y, z) which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**



$$\text{Mass} = \iiint_R \delta \, dV = \int_{y=0}^{y=1} \int_{x=0}^{x=3} \int_{z=0}^{z=\sqrt{34-3x^2-y^2}} e^z(2x+y) \, dz \, dx \, dy$$

Note that on the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 1$, the function $\underline{34 - 3x^2 - y^2} > 0$.