(A)

MAT 1320 A Fall 2015 November 18th, 8:30 Prof. Desjardins

TEST #2

Max = 15

Name: Solutions

Student Number: ____________________

• Time: 80 min.

• Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.

• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

• Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ____________________

A

1. [2 points] Find the derivative of $f(x) = (\cos x)^x$.

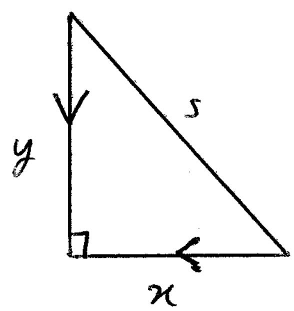
$$\ln(f(x)) = \ln((\cos x)^x) = x \ln \cos x$$

so $$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{d}{dx}\left(x \ln \cos x\right)$$

$$\frac{1}{f(x)} f'(x) = \ln \cos x + x\left(\frac{1}{\cos x}\right)(-\sin x)$$

so $$\boxed{f'(x) = (\cos x)^x\left(\ln \cos x - x \tan x\right)}$$

2. [2 points] A car travelling west at 40 km/h is approaching an intersection. It is currently 2 km away from it. Another car, travelling south at 60 km/h, is 3 km from the same intersection and also moving towards it. At what rate is the distance between the cars changing at this moment?



let x be distance first car is from intersection

let y be distance second car is from intersection

let s be the distance between the cars

told $\frac{dx}{dt} = -40$ km/h, $\frac{dy}{dt} = -60$ km/h

want $\frac{ds}{dt}$ when $x = 2$ and $y = 3$

$s = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$ km

since $s^2 = x^2 + y^2$

and $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

so $$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{\sqrt{13}}\left(2(-40) + 3(-60)\right)$$

$$= \boxed{-\frac{260}{\sqrt{13}} \text{ km/h} \simeq -72 \text{ km/h}}$$

3. [1 point] What is $\frac{d}{dx}\left(\int_1^{x^2} \frac{64t + 5\sin t}{\sqrt{t^3 + e^t + 1}}\, dt\right)$?

$$= \boxed{\frac{2x\left(64x^2 + 5\sin(x^2)\right)}{\sqrt{x^6 + e^{x^2} + 1}}}$$

A

4. [3 points] Evaluate the following integrals:

(a) $\int \sin^2 x \cos^3 x \, dx$ $= \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1-\sin^2 x)\cos x \, dx$

$= \int (\sin^2 x - \sin^4 x)\cos x \, dx = \int (u^2 - u^4)\, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$\left(\text{let } u = \sin x, \ du = \cos x \, dx\right)$ $= \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$

(b) $\int_1^e \left(\frac{8(\ln x)^2}{x} - \frac{2(\ln x)^{1/3}}{x} \right) dx$ $= \int_0^1 (8u^2 - 2u^{1/3})\, du$

$\left(\text{let } u = \ln x, \ du = \frac{1}{x}dx, \ x=1, u=0, \ x=e, u=1\right)$

$= \frac{8}{3}u^3 - \frac{3}{2}u^{4/3} \Big|_0^1$

$= \frac{8}{3}(1)^3 - \frac{3}{2}(1)^{4/3} - 0$

$= \frac{8}{3} - \frac{3}{2} = \boxed{7/6}$

(c) $\int x^2 e^{3x} \, dx$ $= \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3} x e^{3x} dx$

$\left(\text{let } u = x^2, \ du = 2x\,dx, \ dv = e^{3x}dx, \ v = \frac{1}{3}e^{3x}\right)$

$\left(\text{let } u = x, \ du = dx, \ dv = e^{3x}dx, \ v = \frac{1}{3}e^{3x}\right)$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} dx$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left[\frac{1}{3}x e^{3x} - \int \frac{1}{3}e^{3x} dx\right]$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x}\right) + C$

$= \boxed{\frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C}$

5. [2 points] Find $\frac{dy}{dx}$ if $2x^2y^2 + 3xy^3 = 6x$.

$\frac{d}{dx}(2x^2y^2 + 3xy^3) = \frac{d}{dx}(6x)$

$4xy^2 + 4x^2y\frac{dy}{dx} + 3y^3 + 9xy^2\frac{dy}{dx} = 6$

so $(4x^2y + 9xy^2)\frac{dy}{dx} = 6 - 4xy^2 - 3y^3$

thus $\boxed{\frac{dy}{dx} = \frac{6 - 4xy^2 - 3y^3}{4x^2y + 9xy^2}}$

6. [2 points] Find the linear approximation of $f(x) = \sqrt[4]{16+4x}$ at $x=0$ and use it to estimate $\sqrt[4]{20}$.

$$f(0) = \sqrt[4]{16} = 2, \quad f'(x) = \frac{1}{4}(16+4x)^{-3/4}(4) = (16+4x)^{-3/4}$$

$$\text{so } f'(0) = (16)^{-3/4} = \frac{1}{8}$$

$$\text{thus } L(x) = f(0) + f'(0)(x-0) = \boxed{2 + \frac{1}{8}x}$$

$$\text{then } \sqrt[4]{20} = \sqrt[4]{16+4(1)} \approx 2 + \frac{1}{8}(1) = \boxed{\frac{17}{8} = 2.125}$$

7. [2 points] Use R_4 to estimate the area under the curve $y = \dfrac{4}{1+x^2}$ between $x=0$ and $x=1$.

$$n=4 \Rightarrow \Delta x = \frac{1-0}{4} = 0.25, \text{ so } x_0 = 0, \; x_1 = 0.25, \; x_2 = 0.50,$$
$$x_3 = 0.75, \; x_4 = 1$$

$$R_4 = \sum_{j=1}^{4} f(x_j)\Delta x = \frac{1}{4}\left[\frac{4}{1+x_1^2} + \frac{4}{1+x_2^2} + \frac{4}{1+x_3^2} + \frac{4}{1+x_4^2}\right]$$

$$= \frac{1}{1+(0.25)^2} + \frac{1}{1+(0.5)^2} + \frac{1}{1+(0.75)^2} + \frac{1}{1+(1)^2}$$

$$= 0.9411764 + 0.8 + 0.64 + 0.5$$

$$\simeq \boxed{2.8812}$$

8. [1 point] What is the area under the curve $y = \dfrac{4}{1+x^2}$ between $x=0$ and $x=1$?

$$A = \int_0^1 \frac{4}{1+x^2}\,dx = 4\arctan x \Big|_0^1 = 4(\arctan(1) - \arctan(0))$$

$$= 4(\pi/4 - 0)$$

$$= \boxed{\pi} \simeq \boxed{3.1416}$$

(B)

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Signature: ____________________

B

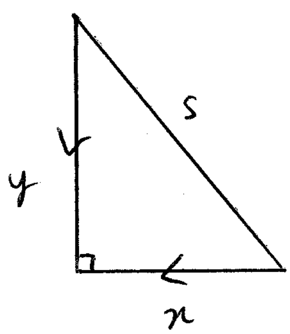
1. [2 points] Find the derivative of $f(x) = (\sin x)^x$.

$$\ln(f(x)) = x \ln(\sin x)$$

so $$\frac{1}{f(x)} f'(x) = \ln \sin x + x\left(\frac{1}{\sin x}\right)(\cos x)$$

$$\therefore \boxed{f'(x) = (\sin x)^x (\ln \sin x + x \cot x)}$$

2. [2 points] A car travelling west at 50 km/h is approaching an intersection. It is currently 2 km away from it. Another car, travelling south at 60 km/h, is 3 km from the same intersection and also moving towards it. At what rate is the distance between the cars changing at this moment?



$$\frac{dx}{dt} = -50 \text{ km/h}, \quad \frac{dy}{dt} = -60 \text{ km/h}$$

$$s^2 = x^2 + y^2, \quad x = 2, \; y = 3, \quad s = \sqrt{13}$$

$$\frac{ds}{dt} = \frac{1}{\sqrt{13}}\left((2)(-50) + 3(-60)\right)$$

$$= \boxed{\frac{-280}{\sqrt{13}} \text{ km/h} \simeq -78 \text{ km/h}}$$

3. [1 point] What is $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2} \frac{16t^2 + \cos t}{\sqrt{t^3 + 7t}}\, dt\right)$?

$$= \boxed{\frac{2x(16x^4 + \cos(x^2))}{\sqrt{x^6 + 7x^2}}}$$

(B)

4. [3 points] Evaluate the following integrals:

(a) $\int \sin^3 x \cos^2 x \, dx$

$$= \int \sin x (1-\cos^2 x)\cos^2 x \, dx$$

$$= \int (\cos^2 x - \cos^4 x) \sin x \, dx \qquad \begin{pmatrix} u = \cos x \\ du = -\sin x \, dx \end{pmatrix}$$

$$= \int (u^4 - u^2)\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$$

(b) $\int_1^e \left(\frac{7(\ln x)^3}{x} - \frac{(\ln x)^{2/3}}{x} \right) dx$

$$= \int_0^1 (7u^3 - u^{2/3})\,du$$

$$= \frac{7}{4}u^4 - \frac{3}{5}u^{5/3} \Big|_0^1$$

$$= \frac{7}{4} - \frac{3}{5} = \boxed{23/20}$$

(c) $\int x^2 e^{4x} \, dx$

$$\begin{pmatrix} u = x^2 \\ du = 2x\,dx \\ dv = e^{4x}dx \\ v = \frac{1}{4}e^{4x} \end{pmatrix}$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int x e^{4x}\,dx \qquad \begin{pmatrix} u = x, \ du = dx \\ dv = e^{4x}dx, \ v = \frac{1}{4}e^{4x} \end{pmatrix}$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[\frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}\,dx\right]$$

$$= \boxed{\frac{1}{4}x^2 e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x} + C}$$

5. [2 points] Find $\frac{dy}{dx}$ if $3x^2y^3 + 4xy^2 = 8x$.

$$6xy^3 + 9x^2y^2\frac{dy}{dx} + 4y^2 + 8xy\frac{dy}{dx} = 8$$

$$\boxed{\frac{dy}{dx} = \frac{8 - 6xy^3 - 4y^2}{9x^2y^2 + 8xy}}$$

B

6. [2 points] Find the linear approximation of $f(x) = \sqrt[3]{27+3x}$ at $x = 0$ and use it to estimate $\sqrt[3]{30}$.

$f(0) = \sqrt[3]{27} = 3, \quad f'(x) = \frac{1}{3}(27+3x)^{-2/3}(3x) = (27+3x)^{-2/3}$

so $f'(0) = (27)^{-2/3} = \frac{1}{9}$

$L(x) = \boxed{3 + \frac{1}{9}x}$

then $\sqrt[3]{30} \approx 3 + \frac{1}{9} = \boxed{28/9 \simeq 3.1111}$

7. [2 points] Use L_4 to estimate the area under the curve $y = \dfrac{4}{1+x^2}$ between $x = 0$ and $x = 1$.

$$L_4 = \sum_{j=0}^{3} f(x_j)\Delta x = \frac{1}{4}\left[\frac{4}{1+x_0^2} + \frac{4}{1+x_1^2} + \frac{4}{1+x_2^2} + \frac{4}{1+x_3^2}\right]$$

$$= \frac{1}{1+(0)^2} + \frac{1}{1+(0.25)^2} + \frac{1}{1+(0.5)^2} + \frac{1}{1+(0.75)^2}$$

$$= 1 + 0.9411764 + 0.8 + 0.64$$

$$\simeq \boxed{3.3812}$$

8. [1 point] What is the area under the curve $y = \dfrac{4}{1+x^2}$ between $x = 0$ and $x = 1$?

$$\text{Area} = \int_0^1 \frac{4}{1+x^2}\,dx = \boxed{\pi}$$

(C)

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Signature: ______________________________

0

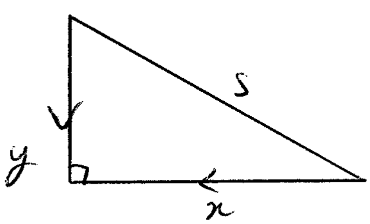
1. [2 points] Find the derivative of $f(x) = x^{\sin x}$.

$$\ln(f(x)) = \sin x \ln x$$

$$\frac{1}{f(x)} f'(x) = \cos x \ln x + \frac{\sin x}{x}$$

So $$\boxed{f'(x) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)}$$

2. [2 points] A car travelling west at 50 km/h is approaching an intersection. It is currently 3 km away from it. Another car, travelling south at 60 km/h, is 2 km from the same intersection and also moving towards it. At what rate is the distance between the cars changing at this moment?



$$\frac{dx}{dt} = -50 \text{ km/h}, \quad \frac{dy}{dt} = -60 \text{ km/h}$$

$$s^2 = x^2 + y^2, \quad x = 3 \quad y = 2, \quad s = \sqrt{13}$$

$$\frac{ds}{dt} = \frac{1}{\sqrt{13}}\left(3(-50) + 2(-60)\right)$$

$$= \boxed{\frac{-270}{\sqrt{13}} \text{ km/h} \simeq -75 \text{ km/h}}$$

3. [1 point] What is $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2} \frac{5t - t\sin t}{\sqrt{t^3 + e^t}}\, dt\right)$?

$$= \boxed{\frac{2x\left(5x^2 - x^2 \sin(x^2)\right)}{\sqrt{x^6 + e^{x^2}}}}$$

c

4. [3 points] Evaluate the following integrals:

(a) $\int \sin^4 x \cos^3 x \, dx$

$$= \int \sin^4 x \,(1-\sin^2 x)\cos x \, dx$$

$$= \int (\sin^4 x - \sin^6 x)\cos x \, dx \qquad \left(\begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array}\right)$$

$$= \int (u^4 - u^6)\, du$$

$$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \boxed{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

(b) $\int_1^e \left(\frac{5(\ln x)^2}{x} - \frac{3(\ln x)^{1/2}}{x}\right) dx$

$$= \int_0^1 (5u^2 - 3u^{1/2})\, du$$

$$= \frac{5}{3}u^3 - 2u^{3/2} \Big|_0^1$$

$$= \frac{5}{3} - 2 = \boxed{-\frac{1}{3}}$$

(c) $\int x^2 e^{2x} \, dx$

$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\, dx \qquad \left(\begin{array}{l} u = x,\ du = dx \\ dv = e^{2x} dx,\ v = \frac{1}{2}e^{2x} \end{array}\right)$$

$$\left(\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ dv = e^{2x}dx \\ v = \frac{1}{2}e^{2x} \end{array}\right) \qquad = \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x e^{2x} - \int \frac{1}{2}e^{2x} dx\right]$$

$$= \boxed{\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C}$$

5. [2 points] Find $\frac{dy}{dx}$ if $4x^3y^2 + 7xy^4 = e^x$.

$$12x^2y^2 + 8x^3y\frac{dy}{dx} + 7y^4 + 28xy^3\frac{dy}{dx} = e^x$$

so $$\boxed{\frac{dy}{dx} = \frac{e^x - 12x^2y^2 - 7y^4}{8x^3y + 28xy^3}}$$

ⓒ

6. [2 points] Find the linear approximation of $f(x) = \sqrt[5]{32+5x}$ at $x = 0$ and use it to estimate $\sqrt[5]{37}$.

$f(0) = \sqrt[5]{32} = 2$, $\quad f'(x) = \frac{1}{5}(32+5x)^{-4/5}(5) = (32+5x)^{-4/5}$

so $f'(0) = (32)^{-4/5} = \frac{1}{16}$

thus $\quad L(x) = \boxed{2 + \frac{1}{16}x}$

and so $\quad \sqrt[5]{37} \approx 2 + \frac{1}{16} = \boxed{33/16 = 2.0625}$

7. [2 points] Use R_4 to estimate the area under the curve $y = \dfrac{2}{1+x^2}$ between $x = 0$ and $x = 1$.

$$R_4 = \sum_{j=1}^{4} f(x_j)\Delta x = \frac{1}{4}\left[\frac{2}{1+x_1^2} + \frac{2}{1+x_2^2} + \frac{2}{1+x_3^2} + \frac{2}{1+x_4^2}\right]$$

$$= \frac{1}{2}\left(\frac{1}{1+(0.25)^2} + \frac{1}{1+(0.5)^2} + \frac{1}{1+(0.75)^2} + \frac{1}{1+(1)^2}\right)$$

$$= \frac{1}{2}(0.9411764 + 0.8 + 0.64 + 0.5)$$

$$\simeq \boxed{1.4406}$$

8. [1 point] What is the area under the curve $y = \dfrac{2}{1+x^2}$ between $x = 0$ and $x = 1$?

$$A = \int_0^1 \frac{2}{1+x^2}\,dx = 2\arctan x\Big|_0^1 = \boxed{\pi/2} \simeq \boxed{1.5708}$$