(8.1) Consider the following first-order ODE:

$$\frac{dy}{dx} = x + y \quad \text{from } x = 0 \text{ to } x = 2.4 \text{ with } y(0) = 2$$

(a) Solve with Euler's explicit method using $h = 0.8$.
(b) Solve with the modified Euler method using $h = 0.8$.
(c) Solve with the classical fourth-order Runge-Kutta method using $h = 0.8$.

The analytical solution of the ODE is $y = 3e^x - x - 1$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Solution

(a) Using Eqs.(8.14) and (8.15) for Euler's explicit method with $h = 0.8$,

$$x_{i+1} = x_i + h = x_i + 0.8$$

$$y_{i+1} = y_i + f(x_i, y_i)h = y_i + (x_i + y_i)0.8$$

$$\text{Error}: \quad E_i = 3e^{x_i} - x_i - 1 - y_i$$

First step: For the first step $i = 1$, $x_1 = 0$, and $y_1 = 2$.

$x_2 = 0 + 0.8 = 0.8$ and $y_2 = y_1 + (x_1 + y_1)0.8 = 2 + (0 + 2)(0.8) = 3.6$

$$E_2 = 3e^{x_2} - x_2 - 1 - y_2 = 3e^{0.8} - 0.8 - 1 - 3.6 = 1.2766$$

Second step: For the second step $i = 2$, $x_2 = 0.8$, and $y_2 = 3.6$

$x_3 = x_2 + 0.8 = 1.6$ and $y_3 = y_2 + (x_2 + y_2)0.8 = 3.6 + (0.8 + 3.6)(0.8) = 7.12$

$$E_3 = 3e^{x_3} - x_3 - 1 - y_3 = 3e^{1.6} - 1.6 - 1 - 7.12 = 5.1391$$

Third step: For the third step, $i = 3$, $x_3 = 1.6$, and $y_3 = 7.12$

$x_4 = x_3 + 0.8 = 2.4$ and $y_4 = y_3 + (x_3 + y_3)0.8 = 7.12 + (1.6 + 7.12)(0.8) = 14.096$

$$E_4 = 3e^{x_4} - x_4 - 1 - y_4 = 3e^{2.4} - 2.4 - 1 - 14.096 = 15.5735$$

(b) Using Eq. (8.56) for the modified Euler method with $h = 0.8$,

$$x_{i+1} = x_i + h = x_i + 0.8$$

$$y_{i+1}^{EU} = y_i + f(x_i, y_i)h = y_i + (x_i + y_i)h$$

$$y_{i+1} = y_i + 0.5[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{EU})]h = y_i + 0.5[x_i + y_i + x_i + y_{i+1}^{EU}]h$$

$$\text{Error}: \quad E_i = 3e^{x_i} - x_i - 1 - y_i$$

First step: For the first step, $i = 1$, $x_1 = 0$, and $y_1 = 2$.

$$x_2 = x_1 + 0.8 = 0.8$$

$$y_2^{EU} = y_1 + (x_1 + y_1)h = 2 + (0 + 2)(0.8) = 3.6$$

$$y_2 = y_1 + 0.5[x_1 + y_1 + x_2 + y_2^{EU}]h = 2 + 0.5[0 + 2 + 0.8 + 3.6](0.8) = 4.56$$

$$E_2 = 3e^{x_2} - x_2 - 1 - y_2 = 3e^{0.8} - 0.8 - 1 - 4.56 = 0.3166$$

Second step: For the second step i = 2, $x_2 = 0.8$, and $y_2 = 4.56$

$$x_3 = x_2 + 0.8 = 1.6$$

$$y_3^{EU} = y_2 + (x_2 + y_2)h = 4.56 + (0.8 + 4.56)(0.8) = 8.848$$

$$y_3 = y_2 + 0.5[x_2 + y_2 + x_3 + y_3^{EU}]h = 4.56 + 0.5[0.8 + 4.56 + 1.6 + 8.848](0.8) = 10.8832$$

$$E_3 = 3e^{x_3} - x_3 - 1 - y_3 = 3e^{1.6} - 1.6 - 1 - 10.8832 = 1.3759$$

Third step: For the third step, i = 3, $x_3 = 1.6$, and $y_3 = 10.8832$

$$x_4 = x_3 + 0.8 = 2.4$$

$$y_4^{EU} = y_3 + (x_3 + y_3)h = 10.8832 + (1.6 + 10.8832)(0.8) = 20.8698$$

$$y_4 = y_3 + 0.5[x_3 + y_3 + x_4 + y_4^{EU}]h = 10.8832 + 0.5[1.6 + 10.8832 + 2.4 + 20.8698](0.8)$$
$$= 25.1844$$

$$E_4 = 3e^{x_4} - x_4 - 1 - y_4 = 3e^{2.4} - 2.4 - 1 - 25.1844 = 4.4851$$

(c) Using Eqs. (8.86) and (8.87) for the classical fourth-order Runge–Kutta method with h = 0.8.

$$x_{i+1} = x_i + h = x_i + 0.8$$

$$f(x_i, y_i) = x_i + y_i$$

$$K_1 = f(x_i, y_i)$$

$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1h)$$

$$K_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_2h)$$

$$K_4 = f(x_i + h, y_i + K_3h)$$

$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$\text{Error}: \quad E_i = 3e^{x_i} - x_i - 1 - y_i$$

First step: For the first step, i = 1, $x_1 = 0$, and $y_1 = 2$.

$$x_2 = x_1 + 0.8 = 0.8$$

$$K_1 = x_1 + y_1 = 0 + 2 = 2$$

$$x_1 + \frac{1}{2}h = 0 + \frac{0.8}{2} = 0.4; \quad y_1 + \frac{1}{2}K_1h = 2 + \frac{(2)(0.8)}{2} = 2.8$$

$$K_2 = 0.4 + 2.8 = 3.2$$

$$x_1 + \frac{h}{2} = 0 + \frac{0.8}{2} = 0.4; \quad y_1 + \frac{1}{2}K_2h = 2 + \frac{(3.2)(0.8)}{2} = 3.28$$

$$K_3 = x_1 + \frac{1}{2}h + y_1 + \frac{1}{2}K_2h = 0 + \frac{0.8}{2} + 3.28 = 3.68$$

$K_4 = x_1 + h + y_1 + K_3h = 0 + 0.8 + 2 + (3.68)(0.8) = 5.744$

$y_2 = 2 + \frac{(0.8)}{6}\left(2 + 2(3.2) + 2(3.68) + 5.744\right) = 4.8672$

Error : $E_2 = 3e^{x_2} - x_2 - 1 - y_2 = 3e^{0.8} - 0.8 - 1 - 4.8672 = 0.0094$

Second step: For the second step i = 2, $x_2 = 0.8$, and $y_2 = 4.8672$

$x_3 = x_2 + 0.8 = 1.6$

$K_1 = x_2 + y_2 = 0.8 + 4.8672 = 5.6672$

$x_2 + \frac{1}{2}h = 0.8 + \frac{0.8}{2} = 1.2; \quad y_2 + \frac{1}{2}K_1h = 4.8672 + \frac{(5.6672)(0.8)}{2} = 7.1341$

$K_2 = x_2 + \frac{1}{2}h + y_2 + \frac{1}{2}K_1h = 1.2 + 7.1341 = 8.3341$

$x_2 + \frac{h}{2} = 0.8 + \frac{0.8}{2} = 1.2; \quad y_2 + \frac{1}{2}K_2h = 4.8672 + \frac{(8.3341)(0.8)}{2} = 8.2008$

$K_3 = x_2 + \frac{1}{2}h + y_2 + \frac{1}{2}K_2h = 0.8 + \frac{0.8}{2} + 8.2008 = 9.4008$

$K_4 = x_2 + h + y_2 + K_3h = 0.8 + 0.8 + 4.8672 + (9.4008)(0.8) = 13.9878$

$$\begin{aligned} y_3 &= y_2 + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) \\ &= 4.8672 + \frac{0.8}{6}(5.6672 + (2)(8.3341) + (2)(9.4008) + 13.9878) \\ &= 12.2172 \end{aligned}$$

Error: $E_3 = 3e^{x_3} - x_3 - 1 - y_3 = 3e^{1.6} - 1.6 - 1 - 12.2172 = 0.0419$

Third step: For the third step, i = 3, $x_3 = 1.6$, and $y_3 = 12.2172$

$x_4 = x_3 + 0.8 = 2.4$

$K_1 = x_3 + y_3 = 1.6 + 12.2172 = 13.8172$

$x_3 + \frac{1}{2}h = 1.6 + \frac{0.8}{2} = 2; \quad y_3 + \frac{1}{2}K_1h = 12.2172 + \frac{(13.8172)(0.8)}{2} = 17.7441$

$K_2 = x_3 + \frac{1}{2}h + y_3 + \frac{1}{2}K_1h = 1.6 + 0.4 + 17.7441 = 19.7441$

$x_3 + \frac{h}{2} = 1.6 + \frac{0.8}{2} = 2; \quad y_3 + \frac{1}{2}K_2h = 12.2172 + \frac{(19.7441)(0.8)}{2} = 20.1148$

$K_3 = x_3 + \frac{1}{2}h + y_3 + \frac{1}{2}K_2h = 1.6 + \frac{0.8}{2} + 20.1148 = 22.1148$

$K_4 = x_3 + h + y_3 + K_3h = 1.6 + 0.8 + 12.2172 + (22.1148)(0.8) = 32.3090$

$$y_4 = y_3 + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$= 12.2172 + \frac{0.8}{6}(13.8172 + (2)(19.7441) + (2)(22.1148) + 32.3090)$$

$$= 29.5297$$

Error: $E_4 = 3e^{x_4} - x_4 - 1 - y_4 = 3e^{2.4} - 2.4 - 1 - 29.5297 = 0.1398$

8.2 Consider the following first-order ODE:

$$\frac{dy}{dx} = 2y - x^2 \qquad \text{from } x = 1 \text{ to } x = 2.8 \text{ with } \; y(1) = 2$$

(*a*) Solve with Euler's explicit method using $h = 0.6$.

(*b*) Solve with the modified Euler method using $h = 0.6$.

(*c*) Solve with the classical fourth-order Runge–Kutta method using $h = 0.6$.

The analytical solution of the ODE is: $y = \frac{1}{2}x + \frac{1}{2}x^2 + \frac{3}{4}e^{2x-2} + \frac{1}{4}$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Solution

(a) Using Eqs.(8.14) and (8.15) for Euler's explicit method with h = 0.6,

$$x_{i+1} = x_i + h = x_i + 0.6$$

$$y_{i+1} = y_i + f(x_i, y_i)h = y_i + (2y_i - x_i^2)0.6$$

$$\text{Error}: \quad E_i = \frac{x_i}{2} + \frac{x_i^2}{2} + \frac{3}{4}e^{2x_i - 2} + \frac{1}{4} - y_i$$

First step: For the first step i = 1, $x_1 = 1$, and $y_1 = 2$.

$x_2 = 1 + 0.6 = 1.6$ and $y_2 = y_1 + (2y_1 - x_1^2)0.6 = 2 + (2(2) - 1)(0.6) = 3.8$

$$E_2 = \frac{1.6}{2} + \frac{(1.6)^2}{2} + \frac{3}{4}e^{2(1.6)-2} + \frac{1}{4} - 3.8 = 4.8201 - 3.8 = 1.0201$$

Second step: For the second step i = 2, $x_2 = 1.6$, and $y_2 = 3.8$

$x_3 = x_2 + 0.6 = 2.2$ and $y_3 = y_2 + (2y_2 - x_2^2)0.6 = 3.8 + (2(3.8) - 2.56)(0.6) = 6.824$

$$E_3 = \frac{2.2}{2} + \frac{(2.2)^2}{2} + \frac{3}{4}e^{2(2.2)-2} + \frac{1}{4} - 6.824 = 12.0374 - 6.824 = 5.2134$$

Third step: For the third step, i = 3, $x_3 = 2.2$, and $y_3 = 6.824$

$x_4 = x_3 + 0.6 = 2.8$ and

$$y_4 = y_3 + (2y_3 - x_3^2)0.6 = 6.824 + (2(6.824) - 4.84)(0.6) = 12.1088$$

$$E_4 = \frac{2.8}{2} + \frac{(2.8)^2}{2} + \frac{3}{4}e^{2(2.8)-2} + \frac{1}{4} - 12.1088 = 33.0187 - 12.1088 = 20.9099$$

(b) Using Eq. (8.56) for the modified Euler method with h = 0.6,

$$x_{i+1} = x_i + h = x_i + 0.6$$

$$y_{i+1}^{EU} = y_i + f(x_i, y_i)h = y_i + (2y_i - x_i^2)h$$

$$y_{i+1} = y_i + 0.5[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{EU})]h = y_i + 0.5[2y_i - x_i^2 + 2y_{i+1}^{EU} - x_{i+1}^2]h$$

$$\text{Error}: \quad E_i = \frac{x_i}{2} + \frac{x_i^2}{2} + \frac{3}{4}e^{2x_i - 2} + \frac{1}{4} - y_i$$

First step: For the first step, $i = 1$, $x_1 = 1$, and $y_1 = 2$.

$$x_2 = x_1 + 0.6 = 1.6$$

$$y_2^{EU} = y_1 + (2y_1 - x_1^2)h = 2 + (2(2) - 1)(0.6) = 3.8$$

$$y_2 = y_1 + 0.5[2y_1 - x_1^2 + 2y_2^{Eu} - x_2^2]h = 2 + 0.5[2(2) - 1 + 2(3.8) - 2.56](0.6) = 4.412$$

$$E_2 = \frac{1.6}{2} + \frac{(1.6)^2}{2} + \frac{3}{4}e^{2(1.6)-2} + \frac{1}{4} - 4.412 = 4.8201 - 4.412 = 0.4081$$

Second step: For the second step $i = 2$, $x_2 = 1.6$, and $y_2 = 4.412$

$$x_3 = x_2 + 0.6 = 2.2$$

$$y_3^{EU} = y_2 + (2y_2 - x_2^2)h = 4.412 + (2(4.412) - 2.56)(0.6) = 8.1704$$

$$y_3 = y_2 + 0.5[2y_2 - x_2^2 + 2y_2^{Eu} - x_3^2]h$$

$$= 4.412 + 0.5[2(4.412) - 2.56 + 2(8.1704) - 4.84](0.6) = 9.7414$$

$$E_3 = \frac{2.2}{2} + \frac{(2.2)^2}{2} + \frac{3}{4}e^{2(2.2)-2} + \frac{1}{4} - 9.7414 = 12.0374 - 9.7414 = 2.296$$

Third step: For the third step, $i = 3$, $x_3 = 2.2$, and $y_3 = 9.7414$

$$x_4 = x_3 + 0.6 = 2.8$$

$$y_4^{EU} = y_3 + (2y_3 - x_3^2)h = 9.7414 + (2(9.7414) - 4.84)(0.6) = 18.5271$$

$$y_4 = y_3 + 0.5[2y_3 - x_3^2 + 2y_4^{Eu} - x_4^2]h$$

$$= 9.7414 + 0.5[2(9.7414) - 4.84 + 2(18.5271) - 7.84](0.6) = 22.8985$$

$$E_3 = \frac{2.8}{2} + \frac{(2.8)^2}{2} + \frac{3}{4}e^{2(2.8)-2} + \frac{1}{4} - 22.8985 = 33.0187 - 22.8985 = 10.1202$$

(c) Using Eqs. (8.86) and (8.87) for the classical fourth-order Runge–Kutta method with $h = 0.6$.

$$x_{i+1} = x_i + h = x_i + 0.6$$

$$f(x_i, y_i) = 2y_i - x_i^2$$

$$K_1 = f(x_i, y_i)$$

$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1h)$$

$$K_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_2h)$$

$$K_4 = f(x_i + h, y_i + K_3h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$\text{Error}: \quad E_i = \frac{x_i}{2} + \frac{x_i^2}{2} + \frac{3}{4}e^{2x_i-2} + \frac{1}{4} - y_i$$

First step: For the first step, i = 1, x_1 = 1, and y_1 = 2.

$$x_2 = x_1 + 0.6 = 1.6$$

$$K_1 = 2y_1 - x_1^2 = 2(2) - 1 = 3$$

$$x_1 + \frac{1}{2}h = 1 + \frac{0.6}{2} = 1.3; \quad y_1 + \frac{1}{2}K_1h = 2 + \frac{(3)(0.6)}{2} = 2.9$$

$$K_2 = 2\left(y_1 + \frac{1}{2}K_1h\right) - \left(x_1 + \frac{1}{2}h\right)^2 = 2\left[2 + \frac{(3)(0.6)}{2}\right] - (1 + 0.3)^2 = 4.11$$

$$x_1 + \frac{h}{2} = 1 + \frac{0.6}{2} = 1.3; \quad y_1 + \frac{1}{2}K_2h = 2 + \frac{(4.11)(0.6)}{2} = 3.233$$

$$K_3 = 2\left(y_1 + \frac{1}{2}K_2h\right) - \left(x_1 + \frac{1}{2}h\right)^2 = 2\left[2 + \frac{(4.11)(0.6)}{2}\right] - (1 + 0.3)^2 = 4.776$$

$$K_4 = 2(y_1 + K_3h) - (x_1 + h)^2 = 2[2 + (4.776)(0.6)] - (1.6)^2 = 7.1712$$

$$y_2 = 2 + \frac{(0.6)}{6}(3 + 2(4.11) + 2(4.776) + 7.1712) = 4.7943$$

$$E_2 = \frac{1.6}{2} + \frac{(1.6)^2}{2} + \frac{3}{4}e^{2(1.6)-2} + \frac{1}{4} - 4.7943 = 4.8201 - 4.7943 = 0.0258$$

Second step: For the second step i = 2, x_2 = 1.6, and y_2 = 4.7943

$$x_3 = x_2 + 0.6 = 2.2$$

$$K_1 = 2y_2 - x_2^2 = 2(4.7943) - 2.56 = 7.0286$$

$$x_2 + \frac{1}{2}h = 1.6 + \frac{0.6}{2} = 1.9; \quad y_2 + \frac{1}{2}K_1h = 4.7943 + \frac{(7.0286)(0.6)}{2} = 6.9029$$

$$K_2 = 2\left(y_2 + \frac{1}{2}K_1h\right) - \left(x_2 + \frac{1}{2}h\right)^2 = 2[6.9029] - (1.9)^2 = 10.1958$$

$$x_2 + \frac{h}{2} = 1.6 + \frac{0.6}{2} = 1.9; \quad y_2 + \frac{1}{2}K_2h = 4.7943 + \frac{(10.1958)(0.6)}{2} = 7.8530$$

$$K_3 = 2\left(y_2 + \frac{1}{2}K_2h\right) - \left(x_2 + \frac{1}{2}h\right)^2 = 2[7.8530] - (1.9)^2 = 12.096$$

$$K_4 = 2(y_2 + K_3h) - (x_2 + h)^2 = 2[4.7943 + (12.096)(0.6)] - (2.2)^2 = 19.2638$$

$$\begin{aligned} y_3 &= y_2 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ &= 4.7943 + \frac{0.6}{6}(7.0286 + (2)(10.1958) + (2)(12.096) + 19.2638) \\ &= 11.8819 \end{aligned}$$

$$E_3 = \frac{2.2}{2} + \frac{(2.2)^2}{2} + \frac{3}{4}e^{2(2.2)-2} + \frac{1}{4} - 11.8819 = 12.0374 - 11.8819 = 0.1555$$

Third step: For the third step, i = 3, $x_3 = 2.2$, and $y_3 = 11.8819$

$$x_4 = x_3 + 0.6 = 2.8$$

$$K_1 = 2y_3 - x_3^2 = 2(11.8819) - 4.84 = 18.9238$$

$$x_3 + \frac{1}{2}h = 2.2 + \frac{0.6}{2} = 2.5; \quad y_3 + \frac{1}{2}K_1h = 11.8819 + \frac{(18.9238)(0.6)}{2} = 17.5590$$

$$K_2 = 2\left(y_3 + \frac{1}{2}K_1h\right) - \left(x_3 + \frac{1}{2}h\right)^2 = 2[17.559] - (2.5)^2 = 28.868$$

$$x_3 + \frac{h}{2} = 2.2 + \frac{0.6}{2} = 2.5; \quad y_3 + \frac{1}{2}K_2h = 11.8819 + \frac{(28.868)(0.6)}{2} = 20.5423$$

$$K_3 = 2\left(y_3 + \frac{1}{2}K_2h\right) - \left(x_3 + \frac{1}{2}h\right)^2 = 2(20.5423) - (2.5)^2 = 34.8346$$

$$K_4 = 2(y_3 + K_3h) - (x_3 + h)^2 = 2[11.8819 + (34.8346)(0.6)] - (2.8)^2 = 57.7253$$

$$y_4 = y_3 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 11.8819 + \frac{0.6}{6}(18.9238 + (2)(28.868) + (2)(34.8346) + 57.7253)$$

$$= 32.2873$$

$$E_4 = \frac{2.8}{2} + \frac{(2.8)^2}{2} + \frac{3}{4}e^{2(2.8)-2} + \frac{1}{4} - 32.2873 = 33.0187 - 32.2873 = 0.7314$$

8.3 Consider the following first-order ODE:

$$\frac{dy}{dt} = t^2 - \frac{3y}{t} \quad \text{from } t = 1 \text{ to } t = 2.2 \text{ with } y(1) = 1$$

(*a*) Solve with Euler's explicit method using $h = 0.4$.
(*b*) Solve with the midpoint method using $h = 0.4$.
(*c*) Solve with the classical fourth-order Runge–Kutta method using $h = 0.4$.

The analytical solution of the ODE is $y = \frac{1}{6}\left(\frac{5}{t^3} + t^3\right)$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Solution

(*a*) The following script file solves the ODE with Euler's explicit method using $h = 0.4$.

```
clear, clc
% Chapter 8se Problem 3 Part (a)
dF=@ (t,y) t.^2-3*y./t;
Fsol=@ (t) (5./t.^3+t.^3)/6;
h=0.4; N=3;
t(1) = 1;  y(1) = 1;
for i = 1:N
    t(i+1) = t(i) + h;
    y(i+1) = y(i) + dF(t(i),y(i))*h;
end
t
y
ys=Fsol(t);
error=ys-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
    1.0000    1.4000    1.8000    2.2000
y =
    1.0000    0.2000    0.8126    1.5669
```

```
error =
         0    0.5610    0.3023    0.2861
```

(*b*) The following script file solves the ODE with modified Euler method using $h = 0.4$.

```
clear, clc
% Chapter 8se Problem 3 Part (b)
dF=@ (t,y) t.^2-3*y./t;
Fsol=@ (t) (5./t.^3+t.^3)/6;
h=0.4; N=3;
t(1) = 1;  y(1) = 1;;
for i = 1:N
    t(i+1) = t(i) + h;
    tm = t(i) + h/2;
    ym=y(i) + dF(t(i),y(i))*h/2;
    y(i+1) = y(i) + (dF(tm,ym))*h;
end
t
y
ys=Fsol(t);
error=ys-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
    1.0000    1.4000    1.8000    2.2000
y =
    1.0000    0.9760    1.2877    1.9838
error =
         0   -0.2150   -0.1728   -0.1309
```

(*c*) The following script file solves the ODE with classical fourth-order Runge–Kutta method using $h = 0.4$.

```
clear, clc
% Chapter 8se Problem 3 Part (c)
dF=@ (t,y) t.^2-3*y./t;
Fsol=@ (t) (5./t.^3+t.^3)/6;
h=0.4; N=3;
t(1) = 1;  y(1) = 1;
for i = 1:N
    t(i+1) = t(i) + h;
    K1= dF(t(i),y(i));
    K2= dF(t(i)+0.5*h,y(i)+K1*h/2);
    K3= dF(t(i)+0.5*h,y(i)+K2*h/2);
    K4= dF(t(i+1),y(i)+K3*h);
    y(i+1) = y(i) + (K1+2*K2+2*K3+K4)*h/6;
end
t
y
xsolp=linspace(t(1),t(N+1),20);
ysolp=Fsol(xsolp);
ys=Fsol(t);
error=ys-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
    1.0000    1.4000    1.8000    2.2000
y =
    1.0000    0.7680    1.1194    1.8559
error =
         0   -0.0070   -0.0045   -0.0030
```

8.4 Consider the following first-order ODE:

$$\frac{dy}{dt} = t^2 - \frac{2y}{t} \quad \text{from} \quad t = 1 \quad \text{to} \quad t = 4 \quad \text{with} \quad y(1) = 2$$

(*a*) Solve with Euler's explicit method using $h = 1$.
(*b*) Solve with the modified Euler method using $h = 1$.
(*c*) Solve with the classical third-order Runge–Kutta method using $h = 1$.

The analytical solution of the ODE is $y = \frac{9}{5t^2} + \frac{t^3}{5}$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Solution

(*a*) The following script file solves the ODE with Euler's explicit method using $h = 1$.

```
clear, clc
% Chapter 8se Problem 4 Part (a)
dF=@ (t,y) t.^2-2*y./t;
Fsol=@ (t) 9./(5*t.^2)+t.^3/5;
h=1; N=3;
t(1) = 1;  y(1) = 2;
for i = 1:N
    t(i+1) = t(i) + h;
    y(i+1) = y(i) + dF(t(i),y(i))*h;
end
t
y
ys=Fsol(t);
error=ys-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
     1     2     3     4
y =
    2.0000   -1.0000    4.0000   10.3333
error =
```

```
         0    3.0500    1.6000    2.5792
```

(*b*) The following script file solves the ODE with modified Euler method using $h = 1$.

```
clear, clc
% Chapter 8se Problem 4 Part (b)
dF=@ (t,y) t.^2-2*y./t;
Fsol=@ (t) 9./(5*t.^2)+t.^3/5;
h=1; N=3;
t(1) = 1;  y(1) = 2;
for i = 1:N
    t(i+1) = t(i) + h;
    K1=dF(t(i),y(i));
    yEU=y(i) + K1*h;
    K2=dF(t(i+1),yEU);
    y(i+1) = y(i) + (K1+K2)/2*h;
end
t
y
ys=Fsol(t);
error=ys-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
     1     2     3     4
y =
    2.0000    3.0000    6.6667   14.1389
error =
         0   -0.9500   -1.0667   -1.2264
```

(*c*) The following script file solves the ODE with classical fourth-order Runge–Kutta method using $h = 1$.

```
clear, clc
% Chapter 8se Problem 4 Part (c)
dF=@ (t,y) t.^2-2*y./t;
Fsol=@ (t) 9./(5*t.^2)+t.^3/5;
h=1; N=3;
t(1) = 1;  y(1) = 2;
for i = 1:N
    t(i+1) = t(i) + h;
    K1= dF(t(i),y(i));
    K2= dF(t(i)+0.5*h,y(i)+K1*h/2);
    K3= dF(t(i)+h,y(i)-K1*h+2*K2*h);
    y(i+1) = y(i) + (K1+4*K2+K3)*h/6;
end
t
y
ys=Fsol(t);
error=ys-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
     1     2     3     4
y =
    2.0000    1.8611    5.4843   12.8245
error =
         0    0.1889    0.1157    0.0880
```

8.5 Consider the following system of two ODEs:

$$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = y - x \quad \text{from } t = 0 \text{ to } t = 2 \text{ with } x(0) = 1\text{, and } y(0) = 2$$

(*a*) Solve with Euler's explicit method using $h = 0.5$.

(*b*) Solve with the modified Euler method using $h = 0.5$.

The analytical solution of the system is $x = e^t[2\sin(t) + \cos(t)]$, $y = e^t[2\cos(t) - \sin(t)]$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Solution

(*a*) The following script file solves the system of the ODEs with Euler's explicit method using $h = 0.5$.

```
clear, clc
dxdt = @ (x,y) x+y;
dydt = @ (x,y) y-x;
x(1) = 1; y(1) = 2;
t(1)=0;
h = 0.5;
N=4;
for i = 1:N
    t(i+1) = t(i) + h;
    x(i+1) = x(i) + dxdt(x(i),y(i))*h;
    y(i+1) = y(i) + dydt(x(i),y(i))*h;
end
t
x
y
Xerror=exp(t).*(2*sin(t)+cos(t))-x
Yerror=exp(t).*(2*cos(t)-sin(t))-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
          0    0.5000    1.0000    1.5000    2.0000
```

```
x =
    1.0000    2.5000    5.0000    8.7500   13.7500
y =
    2.0000    2.5000    2.5000    1.2500   -2.5000
Xerror =
         0    0.5278    1.0434    0.5079   -3.3872
Yerror =
         0   -0.3967   -1.8500   -5.0864  -10.3687
```

(*b*) The following script file solves the system of the ODEs with the modified Euler method using $h = 0.5$.

```
clear, clc
dxdt = @ (x,y) x+y;
dydt = @ (x,y) y-x;
x(1) = 1; y(1) = 2;
t(1)=0;
h = 0.5;
N=4;
for i = 1:N
    t(i+1) = t(i) + h;
    xEU = x(i) + dxdt(x(i),y(i))*h;
    yEU = y(i) + dydt(x(i),y(i))*h;
    x(i+1) = x(i) + (dxdt(x(i),y(i))+dxdt(xEU,yEU))*h/2;
    y(i+1) = y(i) + (dydt(x(i),y(i))+dydt(xEU,yEU))*h/2;
end
t
x
y
Xerror=exp(t).*(2*sin(t)+cos(t))-x
Yerror=exp(t).*(2*cos(t)-sin(t))-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
```

```
         0    0.5000    1.0000    1.5000    2.0000
x =
    1.0000    3.0000    6.1875   10.1250   12.9727
y =
    2.0000    2.2500    1.1250   -2.9531  -12.0234
Xerror =
         0    0.0278   -0.1441   -0.8671   -2.6099
Yerror =
         0   -0.1467   -0.4750   -0.8833   -0.8453
```

8.6 Consider the following system of two ODEs:

$$\frac{dx}{dt} = xt - y \qquad \frac{dy}{dt} = yt + x \qquad \text{from } t = 0 \text{ to } t = 1.2 \text{ with } x(0) = 1\text{, and } y(0) = 0.5$$

(*a*) Solve with Euler's explicit method using $h = 0.4$.

(*b*) Solve with the classical fourth-order Runge–Kutta method using $h = 0.4$.

The analytical solution of the system is $x = e^{\frac{1}{2}t^2}\left(\cos t - \frac{1}{2}\sin t\right)$, $y = \left(-e^{\frac{1}{2}t^2}\right)\left(-\sin t - \frac{1}{2}\cos t\right)$. In each part, calculate the error between the true solution and the numerical solution at the points where the numerical solution is determined.

Solution

(*a*) The following script file solves the system of the ODEs with Euler's explicit method using $h = 0.4$.

```
clear all
dxdt = inline('x*t-y');
dydt = inline('y*t+x');
x(1) = 1; y(1) = 0.5;
t(1)=0;
h = 0.4;
N=3;
for i = 1:N
    t(i+1) = t(i) + h;
    x(i+1) = x(i) + dxdt(t(i),x(i),y(i))*h;
    y(i+1) = y(i) + dydt(t(i),x(i),y(i))*h;
end
t
x
y
Xerror=exp(0.5*t.^2).*(cos(t)-0.5*sin(t))-x
Yerror=-exp(0.5*t.^2).*(-sin(t)-0.5*cos(t))-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
```

```
          0    0.4000    0.8000    1.2000
x =
    1.0000    0.8000    0.5680    0.2042
y =
    0.5000    0.9000    1.3640    2.0277
Xerror =
          0   -0.0132   -0.1025   -0.4171
Yerror =
          0    0.0207    0.1036    0.2594
```

(*b*) The following script file solves the system of the ODEs with the fourth-order Runge-Kutta method using $h = 0.4$.

```
clear all
dxdt = inline('x*t-y');
dydt = inline('y*t+x');
x(1) = 1; y(1) = 0.5;
t(1)=0;
h = 0.4;
N=3;
for i = 1:N
    t(i+1) = t(i) + h;
    Kx1= dxdt(t(i),x(i),y(i));
    Ky1= dydt(t(i),x(i),y(i));
    Kx2= dxdt(t(i)+0.5*h,x(i)+Kx1*h/2,y(i)+Ky1*h/2);
    Ky2= dydt(t(i)+0.5*h,x(i)+Kx1*h/2,y(i)+Ky1*h/2);
    Kx3= dxdt(t(i)+0.5*h,x(i)+Kx2*h/2,y(i)+Ky2*h/2);
    Ky3= dydt(t(i)+0.5*h,x(i)+Kx2*h/2,y(i)+Ky2*h/2);
    Kx4= dxdt(t(i+1),x(i)+Kx3*h,y(i)+Ky3*h);
    Ky4= dydt(t(i+1),x(i)+Kx3*h,y(i)+Ky3*h);
    x(i+1) = x(i) + (Kx1+2*Kx2+2*Kx3+Kx4)*h/6;
    y(i+1) = y(i) + (Ky1+2*Ky2+2*Ky3+Ky4)*h/6;
end
```

```
t
x
y
Xerror=exp(0.5*t.^2).*(cos(t)-0.5*sin(t))-x
Yerror=-exp(0.5*t.^2).*(-sin(t)-0.5*cos(t))-y
```

When the program is executed, the following results are displayed in the Command Window:

```
t =
         0    0.4000    0.8000    1.2000
x =
    1.0000    0.7869    0.4654   -0.2134
y =
    0.5000    0.9206    1.4675    2.2875
Xerror =
  1.0e-003 *
         0   -0.0026    0.1099    0.3956
Yerror =
  1.0e-003 *
         0    0.0922    0.0760   -0.4700
```

8.7 Write the following second-order ODE as a system of two first-order ODEs:

$$\left(\frac{dy}{dt}\right)\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 6y + \sin t = 0$$

Solution

Let $y_1 = y$; $y_2 = \frac{dy_1}{dt} = \frac{dy}{dt}$; This allows the ODE to be re-written as follows:

$$y_2\frac{dy_2}{dt} + 5y_2 - 6y + \sin t = 0$$

$$\text{or } \frac{dy_2}{dt} = \frac{-5y_2 + 6y - \sin t}{y_2}$$

Therefore, the two first order ODEs are:

$\frac{dy_1}{dt} = y_2$, and

$$\frac{dy_2}{dt} = \frac{-5y_2 + 6y - \sin t}{y_2}$$

8.8 Write the following second-order ODEs as systems of two first-order ODEs:

(*a*) $EI\dfrac{d^2y}{dx^2} = -P\sin y + \dfrac{QL}{2}x - \dfrac{Q}{2}x^2$, where *E*, *I*, *P*, *Q*, and *L* are constants.

(*b*) $EI\dfrac{d^2y}{dx^2} = M\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}$, where *E*, *I*, and *M* are constants.

Solution

(*a*) Define $w = \dfrac{dy}{dx}$, $\dfrac{dw}{dx} = \dfrac{d^2y}{dx^2}$. With these definitions, the system of two first-order ODEs is:

$$\frac{dy}{dx} = w$$

$$\frac{dw}{dx} = \frac{1}{EI}\left(-P\sin y + \frac{QL}{2}x - \frac{Q}{2}x^2\right)$$

(*b*) Define $w = \dfrac{dy}{dx}$, $\dfrac{dw}{dx} = \dfrac{d^2y}{dx^2}$. With these definitions, the system of two first-order ODEs is:

$$\frac{dy}{dx} = w$$

$$\frac{dw}{dx} = \frac{M}{EI}[1 + (w)^2]^{3/2}$$

8.9 Write the following second-order ODEs as systems of two first-order ODEs:

(*a*) $\frac{1}{g}\frac{d^2h}{dt^2} = \frac{T}{w} - 1 - \frac{0.008}{w}\left(\frac{dh}{dt}\right)^2$, where g, T, and w are constants.

(*b*) $\frac{d^2Q}{dt^2} + \frac{500}{15}\frac{dQ}{dt} + \frac{250}{15}\left(\frac{dQ}{dt}\right)^3 + \frac{Q}{15 \cdot 4.2 \times 10^{-6}} = \frac{1000}{15}$.

Solution

(*a*) Define $u = \frac{dh}{dt}$, $\frac{du}{dt} = \frac{d^2h}{dt^2}$. With these definitions, the system of two first-order ODEs is:

$$\frac{dh}{dt} = u$$

$$\frac{du}{dt} = \frac{gT}{w} - g - \frac{0.008g}{w}u^2$$

(*b*) Define $u = \frac{dQ}{dt}$, $\frac{du}{dt} = \frac{d^2Q}{dt^2}$. With these definitions, the system of two first-order ODEs is:

$$\frac{dQ}{dt} = u$$

$$\frac{du}{dt} = \frac{1000}{15} - \frac{500}{15}u - \frac{250}{15}u^3 - \frac{Q}{15 \cdot 4.2 \times 10^{-6}}$$

8.10 Consider the following second-order ODE:

$$\frac{d^2y}{dx^2} - 1.8\frac{dy}{dx} + 0.8y = 0 \quad \text{from} \quad x = 0 \quad \text{to} \quad x = 3\text{, with } y(0) = 1 \text{ and } \left.\frac{dy}{dx}\right|_{x=0} = 0.5$$

(*a*) Solve with Euler's explicit method using $h = 1$.
(*b*) Solve with the modified Euler method using $h = 1$.

Solution

Define $w = \frac{dy}{dx}$, $\frac{dw}{dx} = \frac{d^2y}{dx^2}$. With these definitions, the system of two first-order ODEs is:

$$\frac{dy}{dx} = w \quad \text{with} \quad y(0) = 1$$

$$\frac{dw}{dx} = 1.8w - 0.8y \quad \text{with} \quad w(0) = 0.5$$

(*a*) The following script file solves the system of the ODEs with Euler's explicit method using $h = 1$.

```
clear, clc
dydx = @ (w) w;
dwdx = @ (w,y) 1.8*w-0.8*y;
w(1) = 0.5; y(1) = 1;
x(1)=0;
h = 1;
N=3;
for i = 1:N
    x(i+1) = x(i) + h;
    y(i+1) = y(i) + dydx(w(i))*h;
    w(i+1) = w(i) + dwdx(w(i),y(i))*h;
end
x
y
w
```

When the program is executed, the following results are displayed in the Command Window:

```
x =
     0     1     2     3
y =
    1.0000    1.5000    2.1000    2.5800
w =
    0.5000    0.6000    0.4800   -0.3360
```

(*b*) The following script file solves the system of the ODEs with the modified Euler method using $h = 1$.

```
clear, clc
dydx = @ (w) w;
dwdx = @ (w,y) 1.8*w-0.8*y;
w(1) = 0.5; y(1) = 1;
x(1)=0;
h = 1;
N=3;
for i = 1:N
    x(i+1) = x(i) + h;
    yEU = y(i) + dydx(w(i))*h;
    wEU = w(i) + dwdx(w(i),y(i))*h;
    y(i+1) = y(i) + (dydx(w(i))+dydx(wEU))*h/2;
    w(i+1) = w(i) + (dwdx(w(i),y(i))+dwdx(wEU,yEU))*h/2;

end
x
y
w
```

When the program is executed, the following results are displayed in the Command Window:

```
x =
```

```
      0      1      2      3
y =
    1.0000    1.5500    1.8610    0.3828
w =
    0.5000    0.4900   -0.3862   -4.3812
```

8.11 Consider the following second-order ODE:

$$\frac{d^2y}{dx^2} = 1.5(e^{3x} - y^2)^{1/2} \quad \text{from} \quad x = 0 \quad \text{to } x = 1.5, \text{ with } y(0) = 0 \text{ and } \left.\frac{dy}{dx}\right|_{x=0} = 1$$

(*a*) Solve with Euler's explicit method using $h = 0.5$.
(*b*) Solve with the classical fourth-order Runge–Kutta method using $h = 0.5$.

Solution

Define $w = \frac{dy}{dx}$, $\frac{dw}{dx} = \frac{d^2y}{dx^2}$. With these definitions, the system of two first-order ODEs is:

$$\frac{dy}{dx} = w \quad \text{with} \quad y(0) = 0$$

$$\frac{dw}{dx} = 1.5(e^{3x} - y^2)^{1/2} \quad \text{with} \quad w(0) = 1$$

(*a*) The following script file solves the system of the ODEs with Euler's explicit method using $h = 0.5$.

```
clear, clc
dydx = @ (w) w;
dwdx = @ (x,y) 1.5*(exp(3*x)-y^2)^(1/2);
w(1) = 1; y(1) = 0;
x(1)=0;
h = 0.5;
N=3;
for i = 1:N
    x(i+1) = x(i) + h;
    y(i+1) = y(i) + dydx(w(i))*h;
    w(i+1) = w(i) + dwdx(x(i),y(i))*h;
end
x
y
w
```

When the program is executed, the following results are displayed in the Command Window:

```
x =
         0    0.5000    1.0000    1.5000
y =
         0    0.5000    1.3750    3.0214
w =
    1.0000    1.7500    3.2928    6.4920
```

(*b*) The following script file solves the system of the ODEs with the classical fourth-order Runge–Kutta method using $h = 0.5$.

```
clear, clc
dydx = @ (w) w;
dwdx = @ (x,y) 1.5*(exp(3*x)-y^2)^(1/2);
w(1) = 1; y(1) = 0;
x(1)=0;
h = 0.5;
N=3;
for i = 1:N
    x(i+1) = x(i) + h;
    xm = x(i) + h/2;
    Kw1 = dwdx(x(i),y(i));
    Ky1 = dydx(w(i));
    Kw2 = dwdx(xm,y(i)+Ky1*h/2);
    Ky2 = dydx(w(i)+Kw1*h/2);
    Kw3 = dwdx(xm,y(i)+Ky2*h/2);
    Ky3 = dydx(w(i)+Kw2*h/2);
    Kw4 = dwdx(x(i+1),y(i)+Ky3*h);
    Ky4 = dydx(w(i)+Kw3*h);
    w(i+1) = w(i) + (Kw1 + 2*Kw2 + 2*Kw3 + Kw4)*h/6;
    y(i+1) = y(i) + (Ky1 + 2*Ky2 + 2*Ky3 + Ky4)*h/6;
end
```

```
x
y
w
```

When the program is executed, the following results are displayed in the Command Window:

```
x =
         0    0.5000    1.0000    1.5000
y =
         0    0.7404    2.2517    5.2794
w =
    1.0000    2.0833    4.1937    8.4076
```

8.12 Write the following system of two second-order ODEs as a system of four first-order ODEs:

$$\frac{d^2x}{dt^2} = -\frac{\gamma}{m}\left(\frac{dx}{dt}\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \frac{d^2y}{dt^2} = -g - \frac{\gamma}{m}\left(\frac{dy}{dt}\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Solution

Define $w = \frac{dy}{dt}$, $\frac{dw}{dt} = \frac{d^2y}{dt^2}$, and $u = \frac{dx}{dt}$, $\frac{du}{dt} = \frac{d^2x}{dt^2}$

With these definitions, the system of four first-order ODEs is:

$$\frac{dx}{dt} = u \quad \frac{dy}{dt} = w$$

$$\frac{du}{dt} = -\frac{\gamma}{m}u\sqrt{u^2 + w^2} \ , \quad \frac{dw}{dt} = -g - \frac{\gamma}{m}w\sqrt{u^2 + w^2}$$