7.1 The function $f(x)$ is given in the following tabulated form. Compute $\int_0^{1.2} f(x)dx$ with $h = 0.2$ and with $h = 0.4$.
(a) Use the composite rectangle method.
(b) Use the composite trapezoidal method.
(c) Use the composite Simpson's 3/8 method.

x	0	0.2	0.4	0.6	0.8	1.0	1.2
$f(x)$	0.48	0.65	0.87	1.18	1.60	2.15	2.73

Solution

(a) For the composite rectangle method, Eq. (7.4) is used:

$$I(f) = \int_a^b f(x)dx \approx h\sum_{i=1}^{N} f(x_i)$$

There are seven points spaced 0.2 units apart, so if $h = 0.2$, there are six intervals and $N = 6$. In this case, Eq. (7.4) gives

$$I(f) \approx 0.2(0.48 + 0.65 + 0.87 + 1.18 + 1.6 + 2.15)$$
$$I(f) \approx 1.386$$

Note that the last point is not used because the interval between $x = 1.0$ and $x = 1.2$ is approximated by $f(x) = 2.15$.

If $h = 0.4$, there are three intervals (i.e. $N = 3$). In this case, Eq. (7.4) gives

$$I(f) \approx 0.4(0.48 + 0.87 + 1.6)$$
$$I(f) \approx 1.18$$

(b) For the given data, the subintervals are all the same width, so Eq. (7.13) can be used to integrate by the composite trapezoidal method:

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N-1} f(x_i)$$

For $h = 0.2$, there are six intervals and $N = 6$. In this case, Eq. (7.13) gives

$$I(f) \approx \frac{0.2}{2}(0.48 + 2.73) + 0.2(0.65 + 0.87 + 1.18 + 1.6 + 2.15)$$

$$I(f) \approx 1.611$$

For $h = 0.4$, there are three intervals and $N = 3$. In this case, Eq. (7.13) gives

$$I(f) \approx \frac{0.4}{2}(0.48 + 2.73) + 0.4(0.87 + 1.6)$$

$$I(f) \approx 1.63$$

(c) For the given data, the subintervals are all the same width, so Eq. (7.22) can be used to integrate by the composite Simpson's 3/8 method:

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right]$$

For $h = 0.2$, there are six intervals and $N = 6$. In this case, Eq. (7.22) gives

$$I(f) \approx \frac{3 \cdot 0.2}{8}(0.48 + 3(0.65 + 0.87 + 1.6 + 2.15) + 2 \cdot 1.18 + 2.73)$$

$$I(f) \approx 1.6035$$

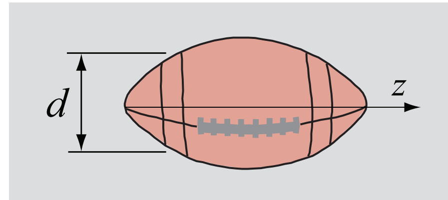
For $h = 0.4$, there are three intervals and $N = 3$. In this case, Eq. (7.22) gives

$$I(f) \approx \frac{3 \cdot 0.4}{8}(0.48 + 3(0.87 + 1.6) + 2.73)$$

$$I(f) \approx 1.593$$

7.2 To estimate the surface area and volume of a football, the diameter of the ball is measured at different points along the ball. The surface area, S, and volume, V, can be determined by:

$$S = 2\pi\int_0^L r dz \quad \text{and} \quad V = \pi\int_0^L r^2 dz$$



Use the data given in the table to determine the volume and surface area of the ball.

z (in.)	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
d (in.)	0	2.6	3.2	4.8	5.6	6	6.2	6.0	5.6	4.8	3.3	2.6	0

(*a*) Use the composite trapezoidal method.
(*b*) Use the composite Simpson's 1/3 method.
(*c*) Use the composite Simpson's 3/8 method.

Solution

(*a*) The composite trapezoidal method is given by Eq. (7.13):

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i)$$

In the present problem, there are 13 points and 12 subintervals.

To calculate the surface area S, The integral $I(r) = \int_0^L r dz$ has to be calculated. First, convert the given data from d to r using $r = \frac{d}{2}$:

z (in.)	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
r (in.)	0	1.3	1.6	2.4	2.8	3.0	3.1	3.0	2.8	2.4	1.65	1.3	0

Next, use Eq. (7.13) to find $I(r)$ for $h = 1.0$ in and $N = 12$:

$$I(r) \approx \frac{1.0}{2}[0 + 0] + 1(1.3 + 1.6 + 2.4 + 2.8 + 3.0 + 3.1 + 3.0 + 2.8 + 2.4 + 1.65 + 1.3)$$

$$I(r) \approx 25.35 \quad \text{in}^2$$

Finally, calculate S using

$$S = 2\pi I(r) \approx 2\pi(25.35)$$

$$S \approx 159.28 \quad \text{in}^2$$

To find the volume *V*, the integral $I(r^2) = \int_0^L r^2 dz$ must be evaluated using Eq. (7.13) using $h = 1.0$ in and $N = 12$:

$$I(r) \approx \frac{1.0}{2}[0^2 + 0^2] + 1(1.3^2 + 1.6^2 + 2.4^2 + 2.8^2 + 3.0^2 + 3.1^2 + 3.0^2 + 2.8^2 + 2.4^2 + 1.65^2 + 1.3^2)$$

$$I(r^2) \approx 63.47 \text{ in}^3$$

Finally, calculate *V* by:

$$V = \pi I(r^2) \approx \pi 63.47 \text{ in}^3$$
$$V \approx 199.4 \quad \text{in}^3$$

(*b*) The composite Simpson's 1/3 method is given by Eq. (7.19):

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

In the present problem, there are 13 points and 12 subintervals. Note that the requirements given for Eq. (7.19) are met. That is, the subintervals are equally spaced and of an even number.

To calculate the surface area *S*, The integral $I(r) = \int_0^L r dz$ has to be calculated. First, convert the given data from *d* to *r* using $r = \frac{d}{2}$:

z (in.)	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
r (in.)	0	1.3	1.6	2.4	2.8	3.0	3.1	3.0	2.8	2.4	1.65	1.3	0

Next, use Eq. (7.19) to find $I(r)$ for $h = 1.0$ in and $N = 12$:

$$I(r) \approx \frac{1.0}{3}[0 + 4(1.3 + 2.4 + 3.0 + 3.0 + 2.4 + 1.3) + 2(1.6 + 2.8 + 3.1 + 2.8 + 1.65) + 0]$$

$$I(r) \approx 25.83 \quad \text{in}^2$$

Finally, calculate *S* using

$$S = 2\pi I(r) \approx 2\pi(25.83)$$
$$S \approx 162.3 \quad \text{in}^2$$

To find the volume V, the integral $I(r^2) = \int_0^L r^2 dz$ is evaluated using Eq. (7.19) with $h = 1.0$ in and $N = 12$:

$$I(r) \approx \frac{1.0}{3}[0^2 + 4(1.3^2 + 2.4^2 + 3.0^2 + 3.0^2 + 2.4^2 + 1.3^2) + 2(1.6^2 + 2.8^2 + 3.1^2 + 2.8^2 + 1.65^2) + 0]$$

$$I(r^2) \approx 64.36 \text{ in}^3$$

Finally, calculate V by:

$$V = \pi I(r^2) \approx \pi 64.25 \text{ in}^3$$

$$V \approx 201.85 \quad \text{in}^3$$

(c) The composite Simpson's 3/8 method is given by Eq. (7.22):

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right]$$

In the present problem, there are 13 points and 12 subintervals. Note that the requirements given for Eq. (7.22) are met. That is, the subintervals are equally spaced and the number of sub intervals is divisible by 3.

To calculate the surface area S, The integral $I(r) = \int_0^L r dz$ has to be calculated. First, convert the given data from d to r using $r = \frac{d}{2}$:

z (in.)	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
r (in.)	0	1.3	1.6	2.4	2.8	3.0	3.1	3.0	2.8	2.4	1.65	1.3	0

Next, use Eq. (7.22) to find $I(r)$ for $h = 1.0$ in and $N = 12$:

$$I(f) \approx \frac{3 \cdot 1}{8}(0 + 3(1.3 + 1.6 + 2.8 + 3 + 3 + 2.8 + 1.65 + 1.3) + 2(2.4 + 3.1 + 2.4) + 0)$$

$$I(r) \approx 25.56 \quad \text{in}^2$$

Finally, calculate S using

$$S = 2\pi I(r) \approx 2\pi(25.56)$$

$$S \approx 160.6 \quad \text{in}^2$$

To find the volume V, the integral $I(r^2) = \int_0^L r^2 dz$ is evaluated using Eq. (7.22) with $h = 1.0$ in and $N = 12$:

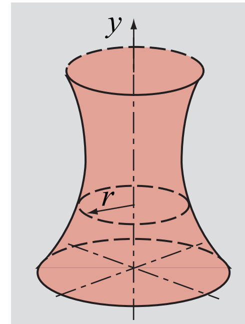
$$I(f) \approx \frac{3 \cdot 1}{8}(0 + 3(1.3^2 + 1.6^2 + 2.8^2 + 3^2 + 3^2 + 2.8^2 + 1.65^2 + 1.3^2) + 2(2.4^2 + 3.1^2 + 2.4^2) + 0)$$

$$I(r^2) \approx 63.48 \text{ in}^3$$

Finally, calculate V by:

$$V = \pi I(r^2) \approx \pi 63.48 \text{ in}^3$$
$$V \approx 199.43 \quad \text{in}^3$$

7.3 To estimate the surface area and volume of a vase, the radius of the vase is measured at different heights. The surface area, S, and volume, V, can be determined by:



$$S = 2\pi\int_0^L r dy \quad \text{and} \quad V = \pi\int_0^L r^2 dy$$

Use the data given below to determine the volume and surface area of the vase:

y (cm)	0	2	4	6	8	10	12	14
r (cm)	15	13.5	11.9	10.8	9.7	9.0	8.4	8.0
y (cm)	16	18	20	22	24	26	28	30
r (cm)	7.8	7.8	7.9	8.0	8.3	8.9	9.6	10.6

(*a*) Use the composite rectangle method.
(*b*) Use the composite trapezoidal method.
(*c*) Use the composite Simpson's 3/8 method.

Solution

(*a*) The composite rectangle method is given by Eq. (7.4):

$$I(f) = \int_a^b f(x)dx \approx h\sum_{i=1}^{N} f(x_i)$$

To calculate the surface area S, The integral $I(r) = \int_0^L r dz$ has to be calculated:

$$I(f) \approx 2(15 + 13.5 + 11.9 + 10.8 + 9.7 + 9 + 8.4 + 8 + 7.8 + 7.8 + 7.9 + 8 + 8.3 + 8.9 + 9.6)$$
$$I(f) \approx 289.2$$

The surface area S is given by:

$$S = 2\pi I(r) \approx 2\pi(289.2)$$
$$S \approx 1817.1 \quad \text{cm}^2$$

To calculate the volume V, The integral $I(r) = \int_0^L r^2 dz$ has to be calculated:

$$I(f) \approx 2(15^2 + 13.5^2 + 11.9^2 + 10.8^2 + 9.7^2 + 9^2 + 8.4^2 + 8^2 + 7.8^2 + 7.8^2 + 7.9^2 + 8^2 + 8.3^2 + 8.9^2 + 9.6^2)$$
$$I(f) \approx 2927$$

The volume V is given by:

$$V = \pi I(r^2) \approx \pi 2927 \ \ \text{in}^3$$
$$9195.4 \quad \text{cm}^3$$

(b) The composite trapezoidal method is given by Eq. (7.13):

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i)$$

To calculate the surface area S, The integral $I(r) = \int_0^L r dz$ has to be calculated:

$$I(r) \approx \frac{2}{2}[15 + 10.6] + 2(13.5 + 11.9 + 10.8 + 9.7 + 9 + 8.4 + 8 + 7.8 + 7.8 + 7.9 + 8 + 8.3 + 8.9 + 9.6)$$

$$I(r) \approx 284.8 \quad \text{cm}^2$$

Finally, calculate S using

$$S = 2\pi I(r) \approx 2\pi(284.8)$$
$$S \approx 1789.5 \quad \text{cm}^2$$

To find the volume V, the integral $I(r^2) = \int_0^L r^2 dz$ must be evaluated:

$$I(r) \approx \frac{2}{2}[15^2 + 10.6^2] + 2(13.5^2 + 11.9^2 + 10.8^2 + 9.7^2 + 9^2 + 8.4^2 + 8^2 + 7.8^2 + 7.8^2 + 7.9^2 + 8^2 + 8.3^2 + 8.9^2 + 9.6^2)$$

$$I(r^2) \approx 2814.4 \ \ \text{cm}^3$$

Finally, calculate V by:

$$V = \pi I(r^2) \approx \pi 2814.4 \quad \text{cm}^3$$
$$V \approx 8841.7 \quad \text{cm}^3$$

(c) The composite Simpson's 3/8 method is given by Eq. (7.22):

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) \ + f(b) \right]$$

In the present problem, there are 13 points and 12 subintervals. Note that the requirements given for Eq. (7.22) are met. That is, the subintervals are equally spaced and the number of sub intervals is divisible by 3.

To calculate the surface area *S*, The integral $I(r) = \int_0^L r dz$ has to be calculated:

$$I(f) \approx \frac{3 \cdot 2}{8}(15 + 3(13.5 + 11.9 + 9.7 + 9 + 8 + 7.8 + 7.9 + 8 + 8.9 + 9.6) + 2(10.8 + 8.4 + 7.8 + 8.3) + 10.6)$$

$$I(r) \approx 284.3 \quad \text{cm}^2$$

Finally, calculate *S* using

$$S = 2\pi I(r) \approx 2\pi(284.3)$$

$$S \approx 1786.3 \quad \text{cm}^2$$

To find the volume *V*, the integral $I(r^2) = \int_0^L r^2 dz$ is evaluated:

$$I(r^2) = \frac{3 \cdot 2}{8}15^2 + 3(13.5^2 + 11.9^2 + 9.7^2 + 9^2 + 8^2 + 7.8^2 + 7.9^2 + 8^2 + 8.9^2 + 9.6^2) +$$

$$2(10.8^2 + 8.4^2 + 7.8^2 + 8.3^2) + 10.6^2)$$

$$I(r^2) \approx 2801.9 \ \text{cm}^3$$

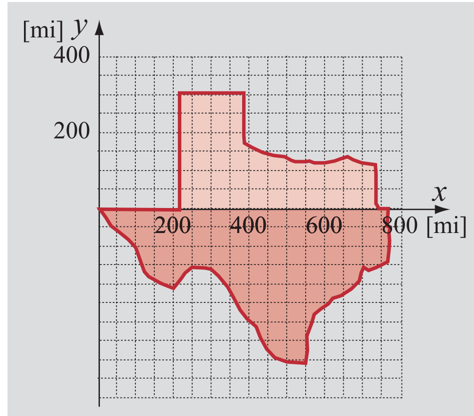
Finally, calculate *V* by:

$$V = \pi I(r^2) \approx \pi 2801.9 \quad \text{cm}^3$$

$$V \approx 8802.4 \quad \text{cm}^3$$

7.4 An approximate map of the state of Texas is shown in the figure. For determining the area of the state, the map is divided into two parts (one above and one below the x axis). Determine the area of the state by numerically integrating the two areas. For each part make a list of the coordinate y of the border as a function of x. Start with $x = 0$ and use increments of 50 mi, such that the last point is $x = 750$.

Once the tabulated data is available, determine the integrals once with the composite trapezoidal method and once with the composite Simpson's 3/8 method.



Solution

The coordinates of the border y at 50-mile increments of x are as follows:

x	0	50	100	150	200	250	300	350	400	450	500
above	0	0	0	0	0	300	300	300	175	150	125
below	0	50	100	175	200	150	150	200	300	375	400

x	550	600	650	700	750
above	125	125	125	125	0
below	400	250	225	150	150

The area of the state is found by integrating the areas above and below the x-axis. Eq. (7.13) is used to integrate using the composite trapezoidal method:

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i)$$

The following MATLAB script is used to calculate the area using Eq. (7.13) with $h = 50$ mi and $N = 15$:

```
clear all; clc;
x=0:50:750;
y_above=[0 0 0 0 0 300 300 300 175 150 125 125 125 125 125 0];
```

```
y_below=[0 50 100 175 200 150 150 200 300 375 400 400 250 225 150 150];
N=length(x)-1;
h=50; %mi, length of subinterval
% Integrate the areas above and below the x-axis using composite midpoint
% method, Eq. (7.13)
for i=1:N
    i_above(i)=(y_above(i)+y_above(i+1))/2;
    i_below(i)=(y_below(i)+y_below(i+1))/2;
end
I_above=h/s*(y_above(1)+y_above(N+1))+h*sum(y_above(2:N));
I_below=h/s*(y_below(1)+y_below(N+1))+h*sum(y_below(2:N));;
% Sum the integrals above and below the x-axis to find the total area
I=I_above+I_below
```

When the above script is executed, the value of $I(f)$ appears in the Command Window:

```
I =
      252500
```

The total area of Texas by the composite midpoint method is $I(f) = 252,500 \text{ mi}^2$.

The composite Simpson's 3/8 method is given by Eq. (7.22):

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right]$$

The requirements for Eq. (7.22) are that the subintervals must be evenly spaced and that their number must be divisible by 3 (see p. 281). There are 15 subintervals spaced at intervals of 50 mi, so the requirements are met. The following MATLAB script is used to calculate the area using Eq. (7.22) with $h = 50$ mi and $N = 15$:

```
clear all; clc;
x=0:50:750;
y_above=[0 0 0 0 0 300 300 300 175 150 125 125 125 125 125 0];
```

```
y_below=[0 50 100 175 200 150 150 200 300 375 400 400 250 225 150 150];
N=length(x)-1;
h=50; %mi, length of subinterval
% Integrate the areas above and below the x-axis using Eq. (7.22)
I_above=3*h/8*(y_above(1)+2*sum(y_above(4:3:(N-2)))+y_above(N+1));
I_above=I_above+3*h/8*3*(sum(y_above(2:3:(N-1)))+sum(y_above(3:3:N)));

I_below=3*h/8*(y_below(1)+2*sum(y_below(4:3:(N-2)))+y_below(N+1));
I_below=I_below+3*h/8*3*(sum(y_below(2:3:(N-1)))+sum(y_below(3:3:N)));

I=I_above+I_below
```

When the above script is executed, the value of $I(f)$ appears in the Command Window:

```
I =
      254060
```

The total area of Texas by the composite Simpson's 3/8 method is $I(f) = 254,060\ \text{mi}^2$.

The area of Texas is actually 261,797 mi^2 (U.S. Census Bureau).

7.5 The speed of a race car during the first six seconds of a race is given by:

t (s)	0	1	2	3	4	5	6
v (mi/h)	0	18	41	63	83	99	112

Determine the distance the car traveled during the first six seconds.
(*a*) Use the composite Trapezoidal method.
(*b*) Use the composite Simpson's 1/3 method.
(*c*) Use the composite Simpson's 3/8 method.

Solution

The distance traveled S is calculated by:

$$S = \int_0^6 v(t)dt$$

(*a*) The composite trapezoidal method is given by Eq. (7.13):

$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i)$$

$$S \approx \left[\frac{1.0}{2}[0 + 112] + 1(18 + 41 + 63 + 83 + 99)\right]\frac{5280}{3600} = 528 \text{ ft}$$

(*b*) The composite Simpson's 1/3 method is given by Eq. (7.19):

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

$$S \approx \frac{1.0}{3}[0 + 4(18 + 63 + 99) + 2(41 + 83) + 112]\frac{5280}{3600} = 528 \text{ ft}$$

(*c*) The composite Simpson's 3/8 method is given by Eq. (7.22):

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right]$$

$$S \approx \frac{3 \cdot 1}{8}(0 + 3(18 + 41 + 83 + 99) + 2(63) + 112)\frac{5280}{3600} = 528.55 \text{ ft}$$

7.6 Evaluate the integral

$$I = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 x dx$$

using the following methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into six subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into six subintervals.

The exact value of the integrals is $I_1 = -4/3$. Compare the results and discuss the reasons for the differences.

Solution

(*a*) The composite Simpson's 1/3 method is given by Eq. (7.19):

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

The integral is calculated in the following script file:

```
clear, clc
N=6;
x=linspace(pi/2,3*pi/2,N+1);
h=(3*pi/2-pi/2)/N;
y=cos(x).^3;
I=h/3*(y(1)+4*sum(y(2:2:N))+2*sum(y(3:2:N-1))+y(N+1))
```

When the script is executed, the following result is displayed in the Command Window:

```
I =
   -1.3261
```

The exact values of the integral is -4/3. The error is less than 1%. The difference is because only 6 subintervals were used in the numerical integration.

(*b*) The composite Simpson's 3/8 method is given by Eq. (7.22):

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i = 2, 5, 8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j = 4, 7, 10}^{N-2} f(x_j) \ + f(b)\right]$$

The integral is calculated in the following script file:

```
clear, clc
N=6;
x=linspace(pi/2,3*pi/2,N+1);
h=(3*pi/2-pi/2)/N;
y=cos(x).^3;
I=3*h/8*(y(1)+3*sum(y(2:3:N-1))+3*sum(y(3:3:N))+2*sum(y(4:3:N-2))+y(N+1))
```

When the script is executed, the following result is displayed in the Command Window:

```
I =
   -1.3052
```

The exact values of the integral is -4/3. The error is less than 3%. The difference is because only 6 subintervals were used in the numerical integration.

7.7 Evaluate the integral

$$I = \int_{-1.2}^{1.2} \frac{1}{1+9x^2}\,dx$$

using the following methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into six subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into six subintervals.

The exact value of the integral is $I = \frac{2\tan^{-1}(18/5)}{3}$. Compare the results and discuss the reasons for the differences.

Solution

(*a*) The composite Simpson's 1/3 method is given by Eq. (7.19):

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

The integral is calculated in the following script file:

```
clear, clc
N=6;
x=linspace(-1.2,1.2,N+1);
h=(1.2-(-1.2))/N;
y=1./(1+9*x.^2);
I=h/3*(y(1)+4*sum(y(2:2:N))+2*sum(y(3:2:N-1))+y(N+1))
```

When the script is executed, the following result is displayed in the Command Window:

```
I =
    0.9288
```

The exact values of the integral is $\frac{2\tan^{-1}(18/5)}{3}$. The error is about 7%. The difference is because only 6 subintervals were used in the numerical integration.

(*b*) The composite Simpson's 3/8 method is given by Eq. (7.22):

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right]$$

The integral is calculated in the following script file:

```
clear, clc
N=6;
x=linspace(-1.2,1.2,N+1);
h=(1.2-(-1.2))/N;
y=1./(1+9*x.^2);
I=3*h/8*(y(1)+3*sum(y(2:3:N-1))+3*sum(y(3:3:N))+2*sum(y(4:3:N-2))+y(N+1))
```

When the script is executed, the following result is displayed in the Command Window:

```
I =
    0.8235
```

The exact values of the integral is $\frac{2\tan^{-1}(18/5)}{3}$. The error is about 5%. The difference is because only 6 subintervals were used in the numerical integration.

7.8 Evaluate the integral from Problem 7.6 using
(*a*) three-point Gauss quadrature;
(*b*) four-point Gauss quadrature.

Solution

The coefficients and Gauss points for second-order Gauss quadrature are given in Table 7-1 and are valid if the range of integration is $[-1, 1]$. Because the range of integration in the present problem is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the integral must be rewritten according to Eq. (7.31):

$$x = \frac{1}{2}[t(b-a)+a+b] = \frac{1}{2}\left[t\left(\frac{3\pi}{2}-\frac{\pi}{2}\right)+\frac{\pi}{2}+\frac{3\pi}{2}\right] = \frac{1}{2}[t\pi+2\pi] = t\frac{\pi}{2}+\pi \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt = \frac{\pi}{2}dt$$

Thus:

$$I = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 x dx = \int_{-1}^{1} \cos^3\left(t\frac{\pi}{2}+\pi\right)\frac{\pi}{2}dt$$

(*a*) The three-point Gauss quadrature formula is:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

where $C_1 = 0.5555556$, $x_1 = -0.77459667$, $C_2 = 0.8888889$, $x_2 = 0$, $C_3 = 0.5555556$, and $x_3 = 0.77459667$. The calculations are done in the following script file:

```
clear, clc
C1=0.5555556; C2=0.8888889; C3=0.5555556;
x1=-0.77459667; x2=0; x3=0.77459667;
F=@ (x) cos(pi/2*x+pi)^3*pi/2;
I=C1*F(x1)+C2*F(x2)+C3*F(x3)
```

When the script is executed, the following answer isplayed in the Command Window:

```
I =
   -1.4690
```

(*b*) The three-point Gauss quadrature formula is:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3) + C_4 f(x_4)$$

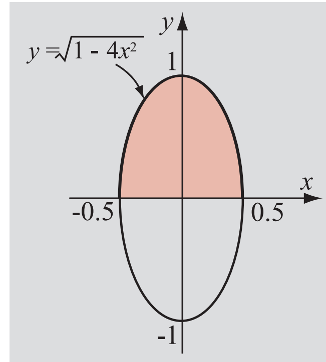
where $C_1 = 0.3478548$, $x_1 = -0.86113631$, $C_2 = 0.6521452$, $x_2 = -0.33998104$, $C_3 = 0.6521452$, $x_3 = 0.33998104$, $C_4 = 0.3478548$ and $x_4 = 0.86113631$. The calculations are done in the following script file:

```
clear, clc
C1=0.3478548; C2=0.6521452; C3=0.6521452; C4=0.3478548;
x1=-0.86113631; x2=-0.33998104; x3=0.33998104; x4=0.86113631;
F=@ (x) cos(pi/2*x+pi)^3*pi/2;
I=C1*F(x1)+C2*F(x2)+C3*F(x3)+C4*F(x4)
```

When the script is executed, the following answer isplayed in the Command Window:

```
I =
   -1.3177
```

7.9 The equation of the ellipse shown in the figure is $4x^2 + y^2 = 1$, and its area is $A = \pi/2$. Consequently, the area of the shaded area is:



$$\int_{-0.5}^{0.5} \sqrt{1-4x^2}\,dx = \frac{\pi}{4}$$

Evaluate the integral using the following methods:

(*a*) Simpson's 1/3 method. Divide the whole interval into eight subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into nine subintervals.
(*c*) Three-point Gauss quadrature.
Compare the results and discuss the reasons for the differences.

Solution

(*a*) For $N = 8$, $h = \frac{b-a}{N} = \frac{1}{8}$.

The composite Simpson's 1/3 method is given by Eq. (7.19):

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

The following MATLAB script is used to calculate I using Eq. (7.19):

```
clear all; clc;
a=-0.5; b=0.5;
N=8;
h=(b-a)/N;
x=a:h:b;
y=sqrt(1-4*x.^2);
% Integrate by Simpon's 1/3 method using Eq. (7.19)
I=h/3*(y(1)+4*sum(y(2:2:N))+2*sum(y(3:2:(N-1)))+y(N+1))
```

When the script is executed, the value of I is displayed in the Command Window:

```
I =
    0.7709
```

The exact value of the integral is $I = \frac{\pi}{4}$, so the error for this method is less than 2%.

(*b*) For $N = 9$, $h = \frac{b-a}{N} = \frac{1}{9}$. Eq. (7.22) is used to calculate the integral using Simpson's 3/8 method:

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) \ + f(b)\right]$$

The following MATLAB script is used to calculate I using Eq. (7.22):

```
clear; clc;
a=-0.5; b=0.5;
N=9;
h=(b-a)/N;
x=a:h:b;
y=sqrt(1-4*x.^2);
% Integrate by Simpon's 3/8 method using Eq. (7.22)
I=3*h/8*(y(1)+3*sum(y(2:3:N-1))+3*sum(y(3:3:N))+2*sum(y(4:3:N-2))+y(N+1))
```

When the script is executed, the value of I is displayed in the Command Window:

```
I =
    0.7706
```

The exact value of the integral is $I = \frac{\pi}{2}$, so the error for this method is less than 2%.

(*c*) The coefficients and Gauss points for second-order Gauss quadrature are given in Table 7-1 and are valid if the range of integration is $[-1, 1]$. Because the range of integration in the present problem is $[-0.5, 0.5]$, the integral must be rewritten according to Eq. (7.31):

$$x = \frac{1}{2}[t(b-a) + a + b] = \frac{1}{2}[t(0.5-(-0.5)) - 0.5 + 0.5] = \frac{1}{2}t \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt = \frac{1}{2}1dt$$

Thus:

$$I = \int_{-0.5}^{0.5} \sqrt{1-4x^2}\,dx = \int_{-1}^{1} \sqrt{1-t^2}\,\frac{1}{2}dt$$

Three-point Gauss quadrature:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

where $C_1 = 0.5555556$, $x_1 = -0.77459667$, $C_2 = 0.8888889$, $x_2 = 0$, $C_3 = 0.5555556$, and $x_3 = 0.77459667$. The calculations are done in the following script file:

```
clear, clc
C1=0.5555556; C2=0.8888889; C3=0.5555556;
x1=-0.77459667; x2=0; x3=0.77459667;
F=@ (x) sqrt(1-x^2)/2;
I=C1*F(x1)+C2*F(x2)+C3*F(x3)
```

When the script is executed, the following answer isplayed in the Command Window:

```
I =
    0.7958
```

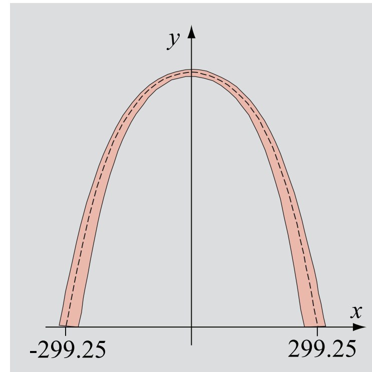
The error for this method is approximately 5%. One would expect the error to decrease if a higher-order Gauss quadrature were to be used.

7.10 The shape of the centroid line of the Gateway Arch in St. Louis can be modeled approximately with the equation:

$$f(x) = 693.9 - 68.8\cosh\left(\frac{x}{99.7}\right) \quad \text{for} \quad -299.25 \le x \le 299.25 \text{ ft.}$$

By using the equation $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$, determine the length of the arch with the following integration methods:

(*a*) Simpson's 1/3 method. Divide the whole interval into eight subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into nine subintervals.
(*c*) Three-point Gauss quadrature.



Solution

(*a*) For $N = 8$,and evenly-spaced points $h = \frac{b-a}{N} = \frac{2(299.25)}{8} = 74.8125$ ft. The derivative of $f(x)$ is:

$$f'(x) = -\frac{68.8}{99.7}\sinh\left(\frac{x}{99.7}\right) = -0.6901\sinh\left(\frac{x}{99.7}\right)$$

The length of the arch is given by:

$$L = \int_a^b \sqrt{1+\left[0.6901\sinh\left(\frac{x}{99.7}\right)\right]^2}\,dx$$

Using the composite Simpson's 1/3 method (Eq. (7.19)) gives:

$$L(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

The last equation is used in the following program (script file) to calculate L.

```
clear all; clc;
a=-299.25; b=299.25;
N=8;
h=(b-a)/N;
x=a:h:b;
fprime=-0.6901.*sinh(x./99.7);
y=sqrt(1+fprime.^2);
% Integrate by Simpon's 1/3 method using Eq. (7.19)
```

```
L=h/3*(y(1)+4*sum(y(2:2:N))+2*sum(y(3:2:(N-1)))+y(N+1))
```

When the script is executed, the value of L is displayed in the Command Window:

```
L =
  1482.3
```

The length of the arch as calculated by Simpson's 1/3 method is $L = 1482.3$ ft.

(b) For $N = 9$ and evenly-spaced points $h = \frac{b-a}{N} = \frac{2(299.25)}{9} = 66.5$ ft.

Eq. (7.22) is used to calculate the length using Simpson's 3/8 method:

$$L(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i\,=\,2,\,5,\,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j\,=\,4,\,7,\,10}^{N-2} f(x_j) \;+ f(b)\right]$$

The last equation is used in the following program (script file) to calculate L.

```
clear all; clc;
a=-299.25; b=299.25;
N=9;
h=(b-a)/N;
x=a:h:b;
fprime=-0.6901.*sinh(x./99.7);
y=sqrt(1+fprime.^2);
% Integrate by Simpon's 3/8 method using Eq. (7.22)
L=3*h/8*(y(1)+2*sum(y(4:3:(N-2)))+y(N+1));
L=I+3*h/8*3*(sum(y(2:3:(N-1)))+sum(y(3:3:N)))
```

When the script is executed, the value of L is displayed in the Command Window:

```
L =
  1483.2
```

The length of the arch as calculated by Simpson's 3/8 method is $L = 1483.2$ ft.

(*c*) The coefficients and Gauss points for second-order Gauss quadrature are given in Table 7-1 and are valid if the range of integration is $[-1, 1]$. Because the range of integration in the present problem is $[-299.25, 299.25]$, the integral must be rewritten according to Eq. (7.31):

$$x = \frac{1}{2}[t(b-a) + a + b] \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt$$

The integration then has the form:

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(b-a)t + a + b}{2}\right)\frac{(b-a)}{2}dt$$

Here, $a = -299.25$ and $b = 299.25$, so

$$L = \int_{-299.25}^{299.25} \sqrt{1 + \left[0.6901 \sinh\left(\frac{x}{99.7}\right)\right]^2}\, dx = \int_{-1}^{1} \sqrt{1 + \left[0.6901 \sinh\left(\frac{598.5t}{199.4}\right)\right]^2}\frac{(598.5)}{2}dt$$

Three-point Gauss quadrature:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

where $C_1 = 0.5555556$, $x_1 = -0.77459667$, $C_2 = 0.8888889$, $x_2 = 0$, $C_3 = 0.5555556$, and $x_3 = 0.77459667$. The calculations are done in the following script file:

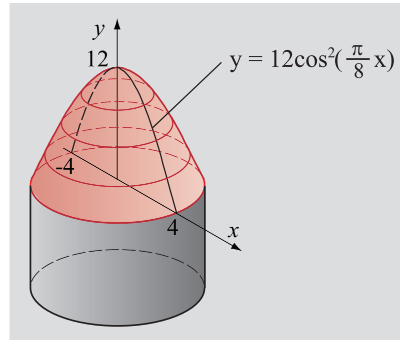
```
clear, clc
C1=0.5555556; C2=0.8888889; C3=0.5555556;
x1=-0.77459667; x2=0; x3=0.77459667;
F=@ (x) sqrt(1+(0.6901*sinh(598.5*x/199.4))^2)*598.5/2;
L=C1*F(x1)+C2*F(x2)+C3*F(x3)
```

When the script is executed, the value of L is displayed in the Command Window:

```
L =
   1.4747e+003
```

The length of the arch as calculated by second-order Gauss quadrature is $L = 1474.7$ ft. It is expected that if a higher-order Gauss quadrature were used, it would yield better agreement with the values of L obtained in parts (*a*) and (*b*).

7.11 The roof of a silo is made by revolving the curve $y = 12\cos^2\left(\frac{\pi}{8}x\right)$ from $x = 0$ m to $x = 4$ m about the y axis, as shown in the figure to the right.



The surface area, S, that is obtained by revolving a curve $y = f(x)$ in the domain from a to b around the y axis can be calculated by:

$$S = 2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}\,dx$$

Calculate the surface area of the roof with the following integration methods:

(a) Simpson's 1/3 method. Divide the whole interval into eight subintervals.
(b) Simpson's 3/8 method. Divide the whole interval into nine subintervals.
(c) Three-point Gauss quadrature.

Solution

(a) The limits of integration are $a = 0$ and $b = 4$. For $N = 8$ and evenly-spaced points $h = \frac{b-a}{N} = \frac{4}{8}$ m.

The derivative of $f(x)$ is:

$$f'(x) = \frac{dy}{dx} = -3\pi\cos\left(\frac{\pi}{8}x\right)\sin\left(\frac{\pi}{8}x\right)$$

The surface area of the roof of the silo is therefore:

$$S = 2\pi\int_0^4 x\sqrt{1 + \left[-3\pi\cos\left(\frac{\pi}{8}x\right)\sin\left(\frac{\pi}{8}x\right)\right]^2}\,dx$$

Using the composite Simpson's 1/3 method (Eq. (7.19)) gives:

$$S(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

The last equation is used in the following program (script file) to calculate S.

```
clear all; clc;
a=0; b=4;
N=8;
h=(b-a)/N;
```

```
x=a:h:b;
fprime=-3*pi*cos(pi/8.*x).*sin(pi/8.*x);
y=x.*sqrt(1+fprime.^2);
% Integrate by Simpon's 1/3 method using Eq. (7.19)
S=h/3*(y(1)+4*sum(y(2:2:N))+2*sum(y(3:2:(N-1)))+y(N+1));
S=S*2*pi
```

When the script is executed, the value of S is displayed in the Command Window:

```
S =
  162.9092
```

The surface area of the roof as calculated by Simpson's 1/3 method is $216.3386\ \text{m}^2$.

(b) For $N = 9$ and evenly-spaced points $h = \frac{b-a}{N} = \frac{4}{9}$ ft.

Eq. (7.22) is used to calculate the surface area using Simpson's 3/8 method:

$$S(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right]$$

The last equation is used in the following program (script file) to calculate S.

```
clear all; clc;
a=0; b=4;
N=9;
h=(b-a)/N;
x=a:h:b;
fprime=-3*pi*cos(pi/8.*x).*sin(pi/8.*x);
y=x.*sqrt(1+fprime.^2);
% Integrate by Simpon's 3/8 method using Eq. (7.22)
S=3*h/8*(y(1)+3*sum(y(2:3:N-1))+3*sum(y(3:3:N))+2*sum(y(4:3:N-2))+y(N+1));
S=2*pi*S
```

When the script is executed, the value of S is displayed in the Command Window:

```
S =
  162.9664
```

The surface area of the silo roof as calculated by Simpson's 3/8 method is 162.9664 m^2.

(c) The coefficients and Gauss points for second-order Gauss quadrature are given in Table 7-1 and are valid if the range of integration is $[-1, 1]$. Because the range of integration in the present problem is $[0, 4]$, the integral must be rewritten according to Eq. (7.31):

$$x = \frac{1}{2}[t(b-a)+a+b] = \frac{1}{2}[t(4-0)+0+4] = 2t+2 \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt = 2dt$$

The integration then has the form:

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(b-a)t+a+b}{2}\right)\frac{(b-a)}{2}dt$$

Here, $a = 0$ and $b = 4$, so

$$S = 2\pi\int_0^4 x\sqrt{1+\left[-3\pi\cos\left(\frac{\pi}{8}x\right)\sin\left(\frac{\pi}{8}x\right)\right]^2}\,dx = 2\pi\int_{-1}^{1}(2t+2)\sqrt{1+\left[-3\pi\cos\left(\frac{\pi}{8}(2t+2)\right)\sin\left(\frac{\pi}{8}(2t+2)\right)\right]^2}\,2\,dt$$

Three-point Gauss quadrature:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

where $C_1 = 0.5555556$, $x_1 = -0.77459667$, $C_2 = 0.8888889$, $x_2 = 0$, $C_3 = 0.5555556$, and $x_3 = 0.77459667$. The calculations are done in the following script file:

```
clear, clc
C1=0.5555556; C2=0.8888889; C3=0.5555556;
x1=-0.77459667; x2=0; x3=0.77459667;
F=@ (x) (2*x+2)*sqrt(1+(-3*pi*cos(pi*(2*x+2)/8)*cos(pi*(2*x+2)/8))^2)*2;
I=C1*F(x1)+C2*F(x2)+C3*F(x3);
S=2*pi*I
```

When the script is executed, the value of S is displayed in the Command Window:

```
S =
  162.3487
```

7.12 In the standard Simpson's 1/3 method (Eq. (7.16)), the points used for the integration are the endpoints of the domain, a and b, and the middle point $(a+b)/2$. Derive a new formula for Simpson's 1/3 method in which the points used for the integration are $x = a$, $x = b$, and $x = (a+b)/3$.

Solution

Starting with the polynomial given by Eq.(7.15),

$$p(x) = f(x_1) + \frac{[f(x_2) - f(x_1)]}{(x_2 - x_1)}(x - x_1) + \left[\frac{f(x_3) - f(x_1)}{(x_3 - x_1)(x_3 - x_2)} - \frac{f(x_2) - f(x_1)}{(x_2 - x_1)(x_3 - x_2)}\right](x - x_1)(x - x_2)$$

Here, $x_1 = a$, $x_2 = (a+b)/3$, and $x_3 = b$. Integrating from $x = a$ to $x = b$ yields:

$$I = f(a)(b-a) + \frac{[f(x_2) - f(a)]}{(x_2 - a)}\left(\frac{b^2}{2} - \frac{a^2}{2} - ab + a^2\right)$$

$$+\left[\frac{f(b) - f(a)}{(b-a)(b-x_2)} - \frac{f(x_2) - f(a)}{(x_2 - a)(b - x_2)}\right]\left\{\frac{b^3}{3} - \frac{a^3}{3} - (a + x_2)\frac{b^2}{2} + \frac{(a + x_2)a^2}{2} + abx_2 - a^2x_2\right\}$$

This can be simplified to yield:

$$I = f(a)(b-a) + \frac{[f(x_2) - f(a)]}{2(x_2 - a)}(b-a)^2$$

$$+\frac{1}{6}\left[\frac{f(b) - f(a)}{(b-a)(b-x_2)} - \frac{f(x_2) - f(a)}{(x_2 - a)(b - x_2)}\right]\{2b^3 + a^3 - 3ab^2 - (a+b)(b-a)^2\}$$

This can be further simplified to yield:

$$I = f(a)(b-a) + \frac{[f(x_2) - f(a)]}{2(x_2 - a)}(b-a)^2$$

$$+\frac{1}{6}\left[\frac{f(b) - f(a)}{(b-a)(b-x_2)} - \frac{f(x_2) - f(a)}{(x_2 - a)(b - x_2)}\right]\{b^3 - 2ab^2 + a^2b\}$$

or,

$$I = f(a)(b-a) + \frac{[f(x_2) - f(a)]}{2(x_2 - a)}(b-a)^2 + \frac{b}{6}\left[\frac{f(b) - f(a)}{(b-a)(b-x_2)} - \frac{f(x_2) - f(a)}{(x_2 - a)(b - x_2)}\right](b-a)^2$$

where $x_2 = (a+b)/3$. Substituting for x_2 yields:

$$I = f(a)(b-a) + \frac{3(b-a)^2}{2(b-2a)}\left[f\left(\frac{a+b}{3}\right) - f(a)\right] + \frac{b(b-a)^2}{2(2b-a)}\left[\frac{[f(b)-f(a)]}{(b-a)} - \frac{3\left[f\left(\frac{a+b}{3}\right) - f(a)\right]}{(b-2a)}\right]$$

7.13 The value of π can be calculated from the integral $\pi = \frac{1}{2}\int_{-1}^{1} \frac{4}{1+x^2}dx$.

(a) Approximate π using the composite trapezoidal method with six subintervals.

(b) Approximate π using the composite Simpson's 1/3 method with six subintervals.

Solution

(a) The composite trapezoidal method is given by Eq. (7.13):

$$I(f) \approx \frac{h}{2}[f(a)+f(b)] + h\sum_{i=2}^{N} f(x_i)$$

The following MATLAB script is used to calculate the integral using Eq. (7.13):

```
clear all; clc;
a=-1; b=1;
N=6;
h=(b-a)/N;
x=a:h:b;
y=4./(1+x.^2);
% Integrate with the composite trapezoidal method using Eq. (7.13)
I=h/2*(y(1)+y(2))+h*sum(y(2:N));
CALpi=I/2
```

When the script is executed, the following result is displayed in the Command Window:

```
CALpi =
    3.1872
```

The composite Simpson's 1/3 method is given by Eq. (7.19):

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right]$$

The following MATLAB script is used to calculate the integral using Eq. (7.13):

```
clear all; clc;
```

```
a=-1; b=1;
N=6;
h=(b-a)/N;
x=a:h:b;
y=4./(1+x.^2);
% Integrate by Simpon's 1/3 method using Eq. (7.19)
I=h/3*(y(1)+4*sum(y(2:2:N))+2*sum(y(3:2:(N-1)))+y(N+1));
CALpi=I/2
```

When the script is executed, the following result is displayed in the Command Window:

```
CALpi =
    3.1419
```

7.14 Evaluate the integral $\int_0^1 225x(1-x^2)\sqrt{(1-x^2)^3}\,dx$ using second-level Romberg integration. Use $n = 1$ in the first estimate with the trapezoidal method.

Solution

For second-level Romberg integration the integral is calculated with the composite trapezoidal integration three times using n, $2n$ and $3n$ subintervals, respectively. After that, the second-level Romberg integration is calculated using the scheme described in Section 7.10:

Subintervals		1st level	2nd level
n	$I_{1,1}$		
		$I_{1,2}$	
$2n$	$I_{2,1}$		$I_{1,3}$
		$I_{2,2}$	
$3n$	$I_{3,1}$		

The first column of integrals ($I_{1,1}$, $I_{2,1}$ and $I_{3,1}$) are calculated using the composite trapezoidal method given by Eq. (7.13):

$$I(f) \approx \frac{h}{2}[f(a)+f(b)] + h\sum_{i=2}^{N-1} f(x_i)$$

The remaining integrals are calculated using Eq. (7.65):

$$I_{i,j} = \frac{4^{j-1}I_{i+1,j-1} - I_{i,j-1}}{4^{j-1}-1} \quad \text{where } j = 2, 3$$

The following MATLAB program (script file) performs the calculations

```
clear all; clc;
a=0; b=1;
n=1;
k=3; % number of columns
for i=1:k
    N=2^(i-1)*n;
    h=(b-a)/N;
```

```
    x=a:h:b;
    y=225*x.*(1-x.^2).*sqrt((1-x.^2).^3);
    % Calculate first column using composite trapezoidal method, Eq. (7.13)
    IR(i,1)=h/2*(y(1)+y(N+1))+h*sum(y(2:N));
end
% Calculate extrapolated values using Romberg method, Eq. (7.65)
for j=2:k
    for i=1:(k-j+1)
        IR(i,j)=(4^(j-1)*IR(i+1,j-1)-IR(i,j-1))/(4^(j-1)-1);
    end
end
IR
```

When the script is executed, the $k \times k$ matrix $[IR]$ is displayed in the Command Window:

```
IR =
         0   36.5354   31.9232
   27.4016   32.2115         0
   31.0090         0         0
```

The value of the integral obtained by second-level Romberg integration is $I = 31.9232$

7.15 Evaluate the integral $53.3904\int_0^{10}(1-e^{-0.18355x})\,dx$ by using four-point Gauss quadrature.

Solution

The following MATLAB script is used to perform the integration:

```
clear all;
a=0;
b=10;
% Coefficients and Gauss points from Table 7-1
C1=0.5555556;
C2=0.8888889;
C3=C1;
t1=-0.77459667;      x(1)=((b-a)*t1+a+b)/2;
t2=0;                x(2)=((b-a)*t2+a+b)/2;
t3=0.77459667;       x(3)=((b-a)*t3+a+b)/2;
% Define F(x)
F_x=(1-exp(-0.18355.*x));

I=C1*F_x(1)+C2*F_x(2)+C3*F_x(3);
I=I*53.3904*(b-a)/2
```

When the script is executed, the value of the integral *I* appears in the Command Window:

```
I =
  289.4362
```

7.16 Evaluate the integral

$$\int_0^1 (-32.9x^5 + 49.4x^4 - 21x^3 + 5.5x^2 - 3.8x + 3.3)\,dx$$

using four-point Gauss quadrature.

Solution

The following MATLAB script is used to perform the integration:

```
clear all; clc;
a=0;
b=1;
% Coefficients and Gauss points from Table 7-1
C1=0.3478548; C2=0.6521452;
C3=C2; C4=C1;
t1=-0.86113631;     x(1)=((b-a)*t1+a+b)/2;
t2=-0.33998104;     x(2)=((b-a)*t2+a+b)/2;
t3=0.33998104;      x(3)=((b-a)*t3+a+b)/2;
t4=0.86113631;      x(4)=((b-a)*t3+a+b)/2;
Fx=-32.9*x.^5+49.4*x.^4-21*x.^3+5.5*x.^2-3.8*x+3.3;
I=C1*Fx(1)+C2*Fx(2)+C3*Fx(3)+C4*Fx(4);
I=I*(b-a)/2
```

When the script is executed, the value of the integral I is displayed in the Command Window:

```
I =
    2.5069
```

7.17 Show that the truncation error for the composite trapezoidal method is of the order of h^2, where h is the step size (width of subinterval).

Solution

The composite trapezoidal method is given by:

$$I(f) = \sum_{i=1}^{N}\int_{x_i}^{x_{i+1}} f(x)dx \tag{1}$$

The trapezoidal rule is derived by expressing the integrand in each subinterval $[x_i, x_{i+1}]$ in a three term Taylor expansion about $x = x_i$:

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \frac{f''(\xi_i)(x - x_i)^2}{2} \tag{2}$$

Setting $x = x_{i+1}$ gives:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(\xi_i)(x_{i+1} - x_i)^2}{2} \tag{3}$$

Solving Eq. (3) for the first derivative:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)} - \frac{f''(\xi_i)(x_{i+1} - x_i)}{2} \tag{4}$$

Substituting (4) into (2) yields:

$$f(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)}(x - x_i) - \frac{f''(\xi_i)(x_{i+1} - x_i)(x - x_i)}{2} + \frac{f''(\xi_i)(x - x_i)^2}{2} \tag{5}$$

Substituting (5) into (1) and performing the integration yields:

$$I(f) = \sum_{i=1}^{N}\left\{f(x_i)(x_{i+1} - x_i) + \frac{[f(x_{i+1}) - f(x_i)](x_{i+1} - x_i)}{2} - \frac{f''(\xi_i)(x_{i+1} - x_i)^3}{4} + \frac{f''(\xi_i)(x - x_i)^3}{6}\right\}$$

or

$$I(f) = \sum_{i=1}^{N}\left\{f(x_i)(x_{i+1} - x_i) + \frac{[f(x_{i+1}) - f(x_i)](x_{i+1} - x_i)}{2} - \frac{f''(\xi_i)(x_{i+1} - x_i)^3}{12}\right\}$$

The first two terms in the series constitute the trapezoidal approximation for the integral, while the last term denotes the total truncation error:

$$E = -\sum_{i=1}^{N}\frac{f''(\xi_i)(x_{i+1} - x_i)^3}{12}$$

For equal subintervals, $(x_{i+1} - x_i) = h$ so that the accrued truncation error is:

$$E = -\sum_{i=1}^{N} \frac{f''(\xi_i)h^3}{12} \tag{6}$$

Now, following the discussion in Section 7.8, the second derivative $f''(\xi_i)$ may be replaced by an average value, analogous to Eq. (7.42):

$$\sum_{i=1}^{N} f''(\xi_i) \approx N\overline{f''} \tag{7}$$

Substituting (7) into (6) and recognizing that $Nh = (b-a)$ yields:

$$E = -\frac{N\overline{f''}h^3}{12} = -\frac{(b-a)h^2\overline{f''}}{12}$$

Thus, the truncation error for the composite trapezoidal rule is of order $O(h^2)$.