4.1 Solve the following system of equations using the Gauss elimination method:
$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_1 + x_2 + x_3 &= 23 \\ x_1 + x_2 - x_3 &= 11 \end{aligned}.$$

Solution

Step 1: Write the system of equations in matrix form:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \\ 11 \end{bmatrix}$$

Using the first element of the matrix as a pivot, $a_{11} = 1$ and $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$.

Step 2: Multiply the first row by $m_{21} = 1$, subtract the result from the second row, and replace the second row with the final result:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \\ 11 \end{bmatrix}$$

Step 3: Next, multiply the first row by $m_{31} = \frac{a'_{31}}{a'_{11}} = \frac{1}{1} = 1$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \\ 2 \end{bmatrix}$$

Step 4: Use the element a'_{22} as the pivot. Multiply the second row by $m_{32} = \frac{a'_{32}}{a'_{22}} = \frac{2}{2} = 1$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \\ -12 \end{bmatrix}$$

Step 4: The matrix is in upper triangular form, apply back-substitution:

$$x_3 = \frac{-12}{-2} = 6$$

$$2x_2 = 14 \quad \Rightarrow \quad x_2 = 7$$

$$x_1 - 7 + 6 = 9 \quad \Rightarrow \quad x_1 = 10$$

Thus, the solution to the given set of simultaneous equations is $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 10 & 7 & 6 \end{bmatrix}$.

4.2 Given the system of equations $[a][x] = [b]$, where $a = \begin{bmatrix} 3 & -2 & 5 \\ 1 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 14 \\ -1 \\ 14 \end{bmatrix}$, determine the solution using the Gauss elimination method.

Solution

Following the same procedure as in Problem 4.1, the row elimination operations proceed as follows:

Step 1: Multiply the first row by $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{3}$, subtract the result from the second row, and replace the second row with the final result:

$$\begin{bmatrix} 3 & -2 & 5 \\ 0 & -\frac{1}{3} & -\frac{5}{3} \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -5\frac{2}{3} \\ 14 \end{bmatrix}$$

Step 2: Multiply the first row by $m_{31} = \frac{a'_{31}}{a'_{11}} = \frac{2}{3}$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 3 & -2 & 5 \\ 0 & -\frac{1}{3} & -\frac{5}{3} \\ & \frac{4}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -5\frac{2}{3} \\ 4\frac{2}{3} \end{bmatrix}$$

Step 3: Use the element a'_{22} as the pivot. Multiply the second row by $m_{32} = \frac{a'_{32}}{a'_{22}} = -4$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 3 & -2 & 5 \\ 0 & -\frac{1}{3} & -\frac{5}{3} \\ & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -5\frac{2}{3} \\ -18 \end{bmatrix}$$

Step 4: Solving by back-substitution, $x_3 = \frac{-18}{-6} = 3$, $-\frac{1}{3}x_2 - \frac{5}{3}3 = -5\frac{2}{3}$ or $x_2 = 2$, and $3x_1 - 2(2) + 5(3) = 14$ or $x_1 = 1$. Thus, the solution is $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

4.3 Consider the following system of two linear equations: $\begin{aligned}0.0003x_1 + 1.566x_2 &= 1.569\\ 0.3454x_1 - 2.436x_2 &= 1.018\end{aligned}$.

(*a*) Solve the system with the Gauss elimination method using rounding with four significant figures.

(*b*) Switch the order of the equations, and solve the system with the Gauss elimination method using rounding with four significant figures.
Check the answers by substituting the solution back in the equations.

Solution

(*a*) First, write the system in matrix form: $\begin{bmatrix}0.0003 & 1.566\\ 0.3454 & -2.436\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1.569\\ 1.018\end{bmatrix}$. Next, apply Gaussian elimination with rounding. With $m_{21} = \frac{a_{21}}{a_{11}} = \frac{0.3454}{0.0003} = 1151$, we have $\begin{bmatrix}0.0003 & 1.566\\ 0.0001 & -1804\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1.569\\ -1805\end{bmatrix}$. Note that the a'_{21} element should be zero but it is not because of the lack of precision. Solving the second equation by back-substitution, $x_2 = \frac{-1805}{-1804} = 1.001$, and $(0.0003)x_1 + 1.566(1.001) = 1.569$ or $x_1 = 3.333$. Substituting these results into the original equations, $\begin{bmatrix}0.0003 & 1.566\\ 0.3454 & -2.436\end{bmatrix}\begin{bmatrix}1.001\\ 3.333\end{bmatrix} = \begin{bmatrix}5.219\\ -7.773\end{bmatrix}$ which is not at all the right hand side that was given in the problem. Consequently, $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1.001\\ 3.333\end{bmatrix}$ is not the correct answer.

(*b*) Switching the order of the equations, we have: $\begin{bmatrix}0.3454 & -2.436\\ 0.0003 & 1.566\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1.018\\ 1.569\end{bmatrix}$. Applying Gaussian elimination with rounding to 4 significant figures yields: $m_{21} = \frac{a_{21}}{a_{11}} = \frac{0.0003}{0.3454} = 0.0008686$ and $\begin{bmatrix}0.3454 & -2.436\\ 0.0000 & 1.568\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1.018\\ 1.570\end{bmatrix}$. Solving by back-substitution, $x_2 = \frac{1.570}{1.568} = 1.001$ and $(0.3454)x_1 - (2.436)(1.001) = 1.018$ or $x_1 = \frac{3.456}{0.3454} = 10.01$. Checking by substituting these solutions into the original system, $\begin{bmatrix} 0.3454 & -2.436 \\ 0.0003 & 1.566 \end{bmatrix} \begin{bmatrix} 10.01 \\ 1.001 \end{bmatrix} = \begin{bmatrix} 1.019 \\ 1.571 \end{bmatrix}$, which is close the original right hand side that was given ($\begin{bmatrix} 1.018 \\ 1.569 \end{bmatrix}$). Thus, the answer of part (*b*) is closer to the correct answer, while that of part (*a*) is completely wrong. This demonstrates the need to perform row exchange operations before pivoting when a matrix is poorly conditioned.

4.4 Solve the following system of equations using the Gauss elimination method.

$$
\begin{aligned}
x_1 + 2x_2 + 2x_3 + x_4 &= 6 \\
2x_1 - 3x_2 + x_3 - 5x_4 &= -5 \\
2x_1 + x_2 - 2x_3 - x_4 &= 0 \\
-x_1 + 2x_2 + x_3 - 2x_4 &= 0
\end{aligned}
$$

Solution

The system of equations in matrix form is:

$$
\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & -3 & 1 & -5 \\ 2 & 1 & -2 & -1 \\ -1 & 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 0 \\ 0 \end{bmatrix}
$$

With $m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$, $m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$, and $m_{41} = \frac{a_{41}}{a_{11}} = \frac{-1}{1} = -1$, the first pass with Gaussian elimination yields:

$$
\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -7 & -3 & -7 \\ 0 & -3 & -6 & -3 \\ 0 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -17 \\ -12 \\ 6 \end{bmatrix}
$$

Next, with $m_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-3}{-7}$, $m_{42} = \frac{a'_{42}}{a'_{22}} = \frac{4}{-7}$, the second pass with Gaussian elimination yields:

$$
\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -7 & -3 & -7 \\ 0 & 0 & -4.7143 & 0 \\ 0 & 0 & 1.2857 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -17 \\ -4.7143 \\ -3.7143 \end{bmatrix}
$$

Next, with $m_{43} = \frac{a''_{43}}{a''_{33}} = \frac{1.2857}{-4.7143}$, the third pass with Gaussian elimination yields:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -7 & -3 & -7 \\ 0 & 0 & -4.7143 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -17 \\ -4.7143 \\ -5 \end{bmatrix}$$

Using back substitution, $x_4 = 1$, $x_3 = \dfrac{-4.7143}{-4.7143} = 1$, $x_2 = \dfrac{-17 + (3)(1) + (7)(1)}{-7} = 1$,

and $x_1 = \dfrac{6 - (2)(1) - (2)(1) - (1)(1)}{1} = 1$. Thus, the solution is: $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.

4.5 Solve the following system of equations with the Gauss elimination method.

$$x_1 + 2x_2 + 4x_3 + 3x_4 = -4$$
$$x_1 - x_2 + x_3 - x_4 = 0$$
$$2x_1 + 5x_2 - 3x_3 - x_4 = 11$$
$$-4x_1 + 2x_2 + x_3 - 4x_4 = 1$$

Solution

The system of equations in matrix form is:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -1 & 1 & -1 \\ 2 & 5 & -3 & -1 \\ -4 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 11 \\ 1 \end{bmatrix}$$

With $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{1} = 1$, $m_{31} = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$, and $m_{41} = \frac{a_{41}}{a_{11}} = \frac{-4}{1} = -4$, the first pass with Gaussian elimination yields:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -3 & -4 \\ 0 & 1 & -11 & -7 \\ 0 & 10 & 17 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 19 \\ -15 \end{bmatrix}$$

Next, with $m_{32} = \frac{a'_{32}}{a'_{22}} = \frac{1}{-3}$, $m_{42} = \frac{a'_{42}}{a'_{22}} = \frac{10}{-3}$, the second pass with Gaussian elimination yields:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -3 & -4 \\ 0 & 0 & -12 & -8.333 \\ 0 & 0 & 7 & -5.333 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 20.333 \\ -1.667 \end{bmatrix}$$

Next, with $m_{43} = \frac{a''_{43}}{a''_{33}} = \frac{7}{-12}$, the third pass with Gaussian elimination yields:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -3 & -4 \\ 0 & 0 & -12 & -8.333 \\ 0 & 0 & 0 & -10.1944 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 20.333 \\ 10.1944 \end{bmatrix}$$

Using back substitution, $x_4 = -1$, $x_3 = \dfrac{20.333 - 8.333(1)}{-12} = -1$, $x_2 = \dfrac{4-3(1)-(4)(1)}{-3} = 1$, and $x_1 = \dfrac{-4-2(1)+(4)(1)+(3)(1)}{1} = 1$. Thus, the solution is: $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$.

4.6 Solve the following system of equations using the Gauss–Jordan method:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 9 \\ 2x_1 + 3x_2 + x_3 &= 23 \\ 3x_1 + 2x_2 - 4x_3 &= 11 \end{aligned}$$

Solution

First, form the augmented matrix, including the right hand side column vector: $\begin{bmatrix} 1 & 2 & -2 & 9 \\ 2 & 3 & 1 & 23 \\ 3 & 2 & -4 & 11 \end{bmatrix}$.

Step 1: The pivot element $a_{11} = 1$ is already 1. Use the first (pivot) row to eliminate the entries below the pivot element:

$$\begin{bmatrix} 1 & 2 & -2 & 9 \\ 2 & 3 & 1 & 23 \\ 3 & 2 & -4 & 11 \end{bmatrix} \begin{matrix} \\ \leftarrow \quad -2\begin{bmatrix} 1 & 2 & -2 & 9 \end{bmatrix} \\ \leftarrow \quad -3\begin{bmatrix} 1 & 2 & -2 & 9 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 2 & -2 & 9 \\ 0 & -1 & 5 & 5 \\ 0 & -4 & 2 & -16 \end{bmatrix}$$

Step 2: Normalize the second row by dividing it by -1:

$$\begin{bmatrix} 1 & 2 & -2 & 9 \\ 0 & 1 & -5 & -5 \\ 0 & -4 & 2 & -16 \end{bmatrix}$$

The pivot element a_{22} is now 1.Use the second (pivot) row to eliminate the entries above and below the pivot element:

$$\begin{bmatrix} 1 & 2 & -2 & 9 \\ 0 & 1 & -5 & -5 \\ 0 & -4 & 2 & -16 \end{bmatrix} \begin{matrix} \leftarrow \quad -2\begin{bmatrix} 0 & 1 & -5 & -5 \end{bmatrix} \\ \\ \leftarrow \quad -(-4)\begin{bmatrix} 0 & 1 & -5 & -5 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & 8 & 19 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & -18 & -36 \end{bmatrix}$$

Step 3: Normalize the third row by dividing it by -18:

$$\begin{bmatrix} 1 & 0 & 8 & 19 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The pivot element a_{33} is now 1.Use the third (pivot) row to eliminate the entries above the pivot element:

$$\begin{bmatrix} 1 & 0 & 8 & 19 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} \leftarrow \\ \leftarrow \\ \end{matrix} \quad \begin{matrix} -8\begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \\ -(-5)\begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \\ \end{matrix} \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

4.7 Solve the system of equations given in Problem 4.2 using the Gauss–Jordan method.

Solution

$$a = \begin{bmatrix} 3 & -2 & 5 \\ 1 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \text{and } b = \begin{bmatrix} 14 \\ -1 \\ 14 \end{bmatrix},$$

First, form the augmented matrix, including the right hand side column vector: $\begin{bmatrix} 3 & -2 & 5 & 14 \\ 1 & -1 & 0 & -1 \\ 2 & 0 & 4 & 14 \end{bmatrix}$.

Step 1: Normalize the first row by dividing it by 3:

$$\begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \\ 1 & -1 & 0 & -1 \\ 2 & 0 & 4 & 14 \end{bmatrix}$$

Use the first (pivot) row to eliminate the entries below the pivot element:

$$\begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \\ 1 & -1 & 0 & -1 \\ 2 & 0 & 4 & 14 \end{bmatrix} \quad \begin{matrix} \\ \leftarrow \\ \leftarrow \end{matrix} \quad \begin{matrix} -1\begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \end{bmatrix} \\ -2\begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \\ 0 & \frac{-1}{3} & \frac{-5}{3} & \frac{-17}{3} \\ 0 & \frac{4}{3} & \frac{2}{3} & \frac{14}{3} \end{bmatrix}$$

Step 2: Normalize the second row by dividing it by $\frac{-1}{3}$:

$$\begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \\ 0 & 1 & 5 & 17 \\ 0 & \frac{4}{3} & \frac{2}{3} & \frac{14}{3} \end{bmatrix}$$

The pivot element a_{22} is now 1.Use the second (pivot) row to eliminate the entries above and below the pivot element:

$$\begin{bmatrix} 1 & \frac{-2}{3} & \frac{5}{3} & \frac{14}{3} \\ 0 & 1 & 5 & 17 \\ 0 & \frac{4}{3} & \frac{2}{3} & \frac{14}{3} \end{bmatrix} \begin{matrix} \leftarrow \\ \\ \leftarrow \end{matrix} \begin{matrix} \left(-\frac{-2}{3}\right)[0\ 1\ 5\ 17] \\ \\ \left(-\frac{4}{3}\right)[0\ 1\ 5\ 17] \end{matrix} = \begin{bmatrix} 1 & 0 & 5 & 16 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & -6 & -18 \end{bmatrix}$$

Step 3: Normalize the third row by dividing it by -6:

$$\begin{bmatrix} 1 & 0 & 5 & 16 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The pivot element a_{33} is now 1.Use the third (pivot) row to eliminate the entries above the pivot element:

$$\begin{bmatrix} 1 & 0 & 5 & 16 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \\ \end{matrix} \begin{matrix} (-5)[0\ 0\ 1\ 3] \\ (-5)[0\ 0\ 1\ 3] \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

4.8 Given the system of equations $[a][x] = [b]$, where $a = \begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$, determine the solution using the Gauss–Jordan method.

Solution

First, form the augmented matrix, including the right hand side column vector: $\begin{bmatrix} 2 & -4 & 1 & 4 \\ 6 & 2 & -1 & 10 \\ -2 & 6 & -2 & -6 \end{bmatrix}$.

Step 1: The pivot element is $a_{11} = 2$. Normalize the first row by dividing it by 2:

$$\begin{bmatrix} 1 & -2 & 0.5 & 2 \\ 6 & 2 & -1 & 10 \\ -2 & 6 & -2 & -6 \end{bmatrix}$$

Use the first (pivot) row to eliminate the entries below the pivot element:

$$\begin{bmatrix} 1 & -2 & 0.5 & 2 \\ 6 & 2 & -1 & 10 \\ -2 & 6 & -2 & -6 \end{bmatrix} \begin{matrix} \\ \leftarrow \quad -6\begin{bmatrix} 1 & -2 & 0.5 & 2 \end{bmatrix} \\ \leftarrow \quad -(-2)\begin{bmatrix} 1 & -2 & 0.5 & 2 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & -2 & 0.5 & 2 \\ 0 & 14 & -4 & -2 \\ 0 & 2 & -1 & -2 \end{bmatrix}$$

Step 2: Normalize the second row by dividing it by 14:

$$\begin{bmatrix} 1 & -2 & 0.5 & 2 \\ 0 & 1 & -0.2857 & -0.1429 \\ 0 & 2 & -1 & -2 \end{bmatrix}$$

The pivot element a_{22} is now 1. Use the second (pivot) row to eliminate the entries above and below the pivot element:

$$\begin{bmatrix} 1 & -2 & 0.5 & 2 \\ 0 & 1 & -0.2857 & -0.1429 \\ 0 & 2 & -1 & -2 \end{bmatrix} \begin{matrix} \leftarrow \quad -(-2)\begin{bmatrix} 0 & 1 & -0.2857 & -0.1429 \end{bmatrix} \\ \\ \leftarrow \quad -2\begin{bmatrix} 0 & 1 & -0.2857 & -0.1429 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & -0.0714 & 1.7143 \\ 0 & 1 & -0.2857 & -0.1429 \\ 0 & 0 & -0.4286 & -1.7143 \end{bmatrix}$$

Step 3: Normalize the third row by dividing it by -0.4286:

$$\begin{bmatrix} 1 & 0 & -0.0714 & 1.7143 \\ 0 & 1 & -0.2857 & -0.1429 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -0.0714 & 1.7143 \\ 0 & 1 & -0.2857 & -0.1429 \\ 0 & 0 & -0.4286 & -1.7143 \end{bmatrix}$$

The pivot element a_{33} is now 1. Use the third (pivot) row to eliminate the entries above the pivot element:

$$\begin{bmatrix} 1 & 0 & -0.0714 & 1.7143 \\ 0 & 1 & -0.2857 & -0.1429 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \end{matrix} \quad \begin{matrix} -(-0.0714)\begin{bmatrix} 0 & 0 & 1 & 4 \end{bmatrix} \\ -(-0.2857)\begin{bmatrix} 0 & 0 & 1 & 4 \end{bmatrix} \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Thus, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

4.9 Solve the following system of equations with the Gauss–Jordan elimination method.

$$5x_1 + 4x_2 + 3x_3 + 2x_4 = -4$$
$$x_1 - 2x_1 + 3x_3 - 4x_4 = 0$$
$$2x_1 + x_2 - x_3 + -2x_4 = -6$$
$$-3x_1 + 4x_2 + 2x_3 - 5x_4 = -4$$

Solution

First, form the augmented matrix, including the right hand side column vector: $\begin{bmatrix} 5 & 4 & 3 & 2 & -4 \\ 1 & -2 & 3 & -4 & 0 \\ 2 & 1 & -1 & -2 & -6 \\ -3 & 4 & 2 & -5 & -4 \end{bmatrix}$

Step 1: The pivot element is $a_{11} = 5$. Normalize the first row by dividing it by 2:

$$\begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & -0.8 \\ 1 & -2 & 3 & -4 & 0 \\ 2 & 1 & -1 & -2 & -6 \\ -3 & 4 & 2 & -5 & -4 \end{bmatrix}$$

Use the first (pivot) row to eliminate the entries below the pivot element:

$$\begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & -0.8 \\ 1 & -2 & 3 & -4 & 0 \\ 2 & 1 & -1 & -2 & -6 \\ -3 & 4 & 2 & -5 & -4 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \quad \begin{matrix} \\ -1\begin{bmatrix}1 & 0.8 & 0.6 & 0.4 & -0.8\end{bmatrix} \\ -2\begin{bmatrix}1 & 0.8 & 0.6 & 0.4 & -0.8\end{bmatrix} \\ -(-3)\begin{bmatrix}1 & 0.8 & 0.6 & 0.4 & -0.8\end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & -0.8 \\ 0 & -2.8 & 2.4 & -4.4 & 0.8 \\ 0 & -0.6 & -2.2 & -2.8 & -4.4 \\ 0 & 6.4 & 3.8 & -3.8 & -6.4 \end{bmatrix}$$

Step 2: Normalize the second row by dividing it by -2.8:

$$\begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & -0.8 \\ 0 & 1 & -0.8571 & 1.5714 & -0.2857 \\ 0 & -0.6 & -2.2 & -2.8 & -4.4 \\ 0 & 6.4 & 3.8 & -3.8 & -6.4 \end{bmatrix}$$

The pivot element a_{22} is now 1. Use the second (pivot) row to eliminate the entries above and below the pivot element:

$$\begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & -0.8 \\ 0 & 1 & -0.8571 & 1.5714 & -0.2857 \\ 0 & -0.6 & -2.2 & -2.8 & -4.4 \\ 0 & 6.4 & 3.8 & -3.8 & -6.4 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \begin{matrix} -0.8\begin{bmatrix}0 & 1 & -0.8571 & 1.5714 & -0.2857\end{bmatrix} \\ \\ -(-0.6)\begin{bmatrix}0 & 1 & -0.8571 & 1.5714 & -0.2857\end{bmatrix} \\ -6.4\begin{bmatrix}0 & 1 & -0.8571 & 1.5714 & -0.2857\end{bmatrix} \end{matrix} =$$

$$\begin{bmatrix} 1 & 0 & 1.2857 & -0.8571 & -0.5714 \\ 0 & 1 & -0.8571 & 1.5714 & 0.2857 \\ 0 & 0 & -2.7143 & -1.8571 & -4.5714 \\ 0 & 0 & 9.2857 & -13.8571 & -4.5714 \end{bmatrix}$$

Step 3: Normalize the third row by dividing it by -2.7143:

$$\begin{bmatrix} 1 & 0 & 1.2857 & -0.8571 & -0.5714 \\ 0 & 1 & -0.8571 & 1.5714 & -0.2857 \\ 0 & 0 & 1 & 0.6842 & 1.6842 \\ 0 & 0 & 9.2857 & -13.8571 & -4.5714 \end{bmatrix}$$

The pivot element a_{33} is now 1. Use the third (pivot) row to eliminate the entries above and below the pivot element:

$$\begin{bmatrix} 1 & 0 & 1.2857 & -0.8571 & -0.5714 \\ 0 & 1 & -0.8571 & 1.5714 & -0.2857 \\ 0 & 0 & 1 & 0.6842 & 1.6842 \\ 0 & 0 & 9.2857 & -13.8571 & -4.5714 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \begin{matrix} -1.2857\begin{bmatrix}0 & 0 & 1 & 0.6842 & 1.6842\end{bmatrix} \\ -(-0.8571)\begin{bmatrix}0 & 0 & 1 & 0.6842 & 1.6842\end{bmatrix} \\ \\ -9.2857\begin{bmatrix}0 & 0 & 1 & 0.6842 & 1.6842\end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & -1.7368 & -2.7368 \\ 0 & 1 & 0 & 2.1579 & 1.1578 \\ 0 & 0 & 1 & 0.6842 & 1.6842 \\ 0 & 0 & 0 & -20.2105 & -20.2105 \end{bmatrix}$$

Step 4: Normalize the fourth row by dividing it by -20.2105:

$$\begin{bmatrix} 1 & 0 & 0 & -1.7368 & -2.7368 \\ 0 & 1 & 0 & 2.1579 & 1.1578 \\ 0 & 0 & 1 & 0.6842 & 1.6842 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The pivot element a_{44} is now 1. Use the third (pivot) row to eliminate the entries above the pivot element:

$$\begin{bmatrix} 1 & 0 & 0 & -1.7368 & -2.7368 \\ 0 & 1 & 0 & 2.1579 & 1.1578 \\ 0 & 0 & 1 & 0.6842 & 1.6842 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \begin{matrix} -(-1.7368)\begin{bmatrix}0 & 0 & 0 & 1 & 1\end{bmatrix} \\ -2.1579\begin{bmatrix}0 & 0 & 0 & 1 & 1\end{bmatrix} \\ -0.6841\begin{bmatrix}0 & 0 & 0 & 1 & 1\end{bmatrix} \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

4.10 Determine the LU decomposition of the matrix $a = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ using the Gauss elimination procedure.

Solution

LU decomposition using Gausian elimination transforms the above matrix into a lower triangular matrix $[L]$ multiplied by an upper triangular matrix $[U]$. $[U]$ is the upper triangular matrix that would normally result after applying Gaussian elimination to the given matrix. $[L]$ consists of the multipliers that are used in the Gaussian elimination procedure and 1s along the diagonal. Thus, with $m_{21} = \frac{2}{4}$ and $m_{31} = \frac{1}{4}$,

$\begin{bmatrix} 4 & 5 & 6 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 5 & 6 \\ 0 & -1.5 & 0 \\ 0 & 0.75 & 1.5 \end{bmatrix}$. Next, with $m_{32} = 0.75/(-1.5) = -0.5$, $\begin{bmatrix} 4 & 5 & 6 \\ 0 & -1.5 & 0 \\ 0 & 0.75 & 1.5 \end{bmatrix} \to \begin{bmatrix} 4 & 5 & 6 \\ 0 & -1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$. Thus,

$U = \begin{bmatrix} 4 & 5 & 6 \\ 0 & -1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$, and $L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{-1}{2} & 1 \end{bmatrix}$. Therefore, $\begin{bmatrix} 4 & 5 & 6 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{-1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 0 & -1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$.

4.11 Determine the LU decomposition of the matrix $a = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{bmatrix}$ using Crout's method.

Solution

The LU decomposition is done by following the procedure described in Section 4.5.2.

$L_{11} = a_{11} = 1$, $U_{12} = \dfrac{a_{12}}{L_{11}} = 2$, $U_{13} = 3$, $L_{21} = a_{21} = 4$, $L_{22} = a_{22} - L_{21}U_{12} = 5 - (4)(2) = -3$,

$L_{31} = a_{31} = 3$, $U_{23} = \dfrac{a_{23} - L_{21}U_{13}}{L_{22}} = \dfrac{6 - (4)(3)}{-3} = 2$, $L_{32} = a_{32} - L_{31}U_{12} = 2 - (3)(2) = -4$,

$L_{33} = a_{33} - L_{31}U_{13} - L_{32}U_{23} = 2 - (3)(3) - (-4)(2) = 1$. Thus, $[L] = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 3 & -4 & 1 \end{bmatrix}$ and $[U] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. To check the answer, multiply the two matrices to see if the original matrix is obtained:

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{bmatrix} = [a]$$

4.12 Solve the following system with LU decomposition using Crout's method.

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

Solution

First the LU decomposition of the matrix of coefficients is done by following the procedure described in Section 4.5.2.

$L_{11} = a_{11} = 2$, $\quad U_{12} = \dfrac{a_{12}}{L_{11}} = -\dfrac{4}{2} = -2$, $\quad U_{13} = \dfrac{a_{13}}{L_{11}} = \dfrac{1}{2} = 0.5$, $\quad L_{21} = a_{21} = 6$,

$L_{22} = a_{22} - L_{21}U_{12} = 2 - 6(-2) = 14$, $\quad L_{31} = a_{31} = -2$, $\quad U_{23} = \dfrac{a_{23} - L_{21}U_{13}}{L_{22}} = \dfrac{-1-(6)(0.5)}{14} = -0.2857$,

$L_{32} = a_{32} - L_{31}U_{12} = -2-(-2)(-2) = 2$,

$L_{33} = a_{33} - L_{31}U_{13} - L_{32}U_{23} = -2-(-2)(0.5) - 2(-0.2857) = -0.4286$. Thus, $[L] = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 14 & 0 \\ -2 & 2 & -0.4286 \end{bmatrix}$ and

$[U] = \begin{bmatrix} 1 & -2 & 0.5 \\ 0 & 1 & -0.2857 \\ 0 & 0 & 1 \end{bmatrix}$. To check the answer, multiply the two matrices to see if the original matrix is obtained:

$$[L][U] = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 14 & 0 \\ -2 & 2 & -0.4286 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0.5 \\ 0 & 1 & -0.2857 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} = [a]$$

Next, $[L]$ and $[b]$ are substituted in Eq. (4.23):

$$\begin{bmatrix} 2 & 0 & 0 \\ 6 & 14 & 0 \\ -2 & 2 & -0.4286 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

This equation is solved for $[y]$ by using forward substitution:

$$x' = 2\,, \quad y' = \frac{10-(6\cdot 2)}{14} = -0.1429\,, \text{ and} \qquad z' = \frac{-6-(-2\cdot 2)-(2\cdot -0.1429)}{-0.4286} = 4 \qquad .$$

Next $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ is substituted in Eq. (4.22):

$$\begin{bmatrix} 1 & -2 & 0.5 \\ 0 & 1 & -0.2857 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -0.1429 \\ 4 \end{bmatrix}$$

The last equation is solved for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ using back substitution

$$z = 4\,, \quad y = \frac{-0.1429-(-0.2857\cdot 4)}{1} = 1 \quad , \text{and } x = \frac{2-(-2\cdot 1)-(0.5\cdot 4)}{1} = 2$$

The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

Check the answer by substituting into the original system:

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

which matches the original right hand side.

4.13 Find the inverse of the matrix $\begin{bmatrix} -4/5 & -3/5 & -2/5 \\ -3/5 & -6/5 & -4/5 \\ -2/5 & -4/5 & -6/5 \end{bmatrix}$ using the Gauss–Jordan method.

Solution

First form the augmented matrix with the identity matrix:

$$\begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ -\frac{3}{5} & -\frac{6}{5} & -\frac{4}{5} & 0 & 1 & 0 \\ -\frac{2}{5} & -\frac{4}{5} & -\frac{6}{5} & 0 & 0 & 1 \end{bmatrix}$$

Step 1: The pivot element is $a_{11} = -\frac{4}{5}$. Normalize the first row by dividing it by $-\frac{4}{5}$:

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\ -\frac{3}{5} & -\frac{6}{5} & -\frac{4}{5} & 0 & 1 & 0 \\ -\frac{2}{5} & -\frac{4}{5} & -\frac{6}{5} & 0 & 0 & 1 \end{bmatrix}$$

The pivot element a_{11} is now 1. Use the first (pivot) row to eliminate the entries below the pivot element:

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\ -\frac{3}{5} & -\frac{6}{5} & -\frac{4}{5} & 0 & 1 & 0 \\ -\frac{2}{5} & -\frac{4}{5} & -\frac{6}{5} & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \quad -\left(-\frac{3}{5}\right)\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \end{bmatrix} \\ \leftarrow \quad -\left(-\frac{2}{5}\right)\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\ 0 & -\frac{3}{4} & -\frac{1}{2} & -\frac{3}{4} & 1 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

Step 2: The pivot element is $a_{22} = -\frac{3}{4}$. Normalize the second row by dividing it by $-\frac{3}{4}$:

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{3} & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

The pivot element a_{22} is now 1. Use the second (pivot) row to eliminate the entries below and above the pivot element:

$$\begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & -\frac{5}{4} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{3} & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow & -\left(\frac{3}{4}\right)\begin{bmatrix} 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{4} & 0 \end{bmatrix} \\ \\ \leftarrow & -\left(-\frac{1}{2}\right)\begin{bmatrix} 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{4} & 0 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & -\frac{2}{3} & 1 \end{bmatrix}$$

Step 3: The pivot element is $a_{33} = -\frac{2}{3}$. Normalize the third row by dividing it by $-\frac{2}{3}$:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

The pivot element a_{33} is now 1. Use the third (pivot) row to eliminate the entries above the pivot element:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \begin{matrix} \leftarrow & -(-2)\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \\ \leftarrow & -\frac{2}{3}\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

The inverse of the matrix is then:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

4.14 Given the matrix $a = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$, determine the inverse of $[a]$ using the Gauss–Jordan method.

Solution

First form the augmented matrix with the identity matrix:

$$\begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 1 \end{bmatrix}$$

Step 1: The pivot element is $a_{11} = -2$. Normalize the first row by dividing it by -2 :

$$\begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 1 \end{bmatrix}$$

The pivot element a_{11} is now 1. Use the first (pivot) row to eliminate the entries below the pivot element:

$$\begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \\ \leftarrow \end{matrix} \begin{matrix} \\ -1\begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \end{bmatrix} \\ -0\begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \end{bmatrix} \end{matrix} = \begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \\ 0 & -1.5 & 1 & 0.5 & 1 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: The pivot element is $a_{22} = -1.5$. Normalize the second row by dividing it by -1.5 :

$$\begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \\ 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 1 \end{bmatrix}$$

The pivot element a_{22} is now 1. Use the second (pivot) row to eliminate the entries below and above the pivot element

$$\begin{bmatrix} 1 & -0.5 & 0 & -0.5 & 0 & 0 \\ 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow \\ \\ \leftarrow \end{matrix} \begin{matrix} -(-0.5)\begin{bmatrix} 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \end{bmatrix} \\ \\ -(1)\begin{bmatrix} 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \end{bmatrix} \end{matrix} =$$

$$\begin{bmatrix} 1 & 0 & -0.3333 & -0.6667 & -0.3333 & 0 \\ 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \\ 0 & 0 & -0.8333 & 0.3333 & 0.66667 & 1 \end{bmatrix}$$

Step 3: The pivot element is $a_{33} = -0.8333$. Normalize the third row by dividing it by -0.8333:

$$\begin{bmatrix} 1 & 0 & -0.3333 & -0.6667 & -0.3333 & 0 \\ 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \\ 0 & 0 & 1 & -0.4 & -0.8 & -1.2 \end{bmatrix}$$

The pivot element a_{33} is now 1. Use the third (pivot) row to eliminate the entries above the pivot element:

$$\begin{bmatrix} 1 & 0 & -0.3333 & -0.6667 & -0.3333 & 0 \\ 0 & 1 & -0.6667 & -0.3333 & -0.6667 & 0 \\ 0 & 0 & 1 & -0.4 & -0.8 & -1.2 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \\ \end{matrix} \begin{matrix} -(-0.3333)\begin{bmatrix} 0 & 0 & 1 & -0.4 & -0.8 & -1.2 \end{bmatrix} \\ -(-0.6667)\begin{bmatrix} 0 & 0 & 1 & -0.4 & -0.8 & -1.2 \end{bmatrix} \\ \\ \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & -0.8 & -0.6 & -0.4 \\ 0 & 1 & 0 & -0.6 & -1.2 & -0.8 \\ 0 & 0 & 1 & -0.4 & -0.8 & -1.2 \end{bmatrix}$$

The inverse of the matrix is then:

$$\begin{bmatrix} -0.8 & -0.6 & 0.4 \\ -0.6 & -1.2 & -0.8 \\ -0.4 & -0.8 & -1.2 \end{bmatrix}$$

4.15 Carry out the first three iterations of the solution of the following system of equations using the Gauss–Seidel iterative method. For the first guess of the solution, take the value of all the unknowns to be zero.

$$8x_1 + 2x_2 + 3x_3 = 51$$
$$2x_1 + 5x_2 + x_3 = 23$$
$$-3x_1 + x_2 + 6x_3 = 20$$

Solution

The essence of the Gauss-Seidel iterative method is given by Eq. (4.51):

$$x_i = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1 \\ j \neq i}}^{j=n} a_{ij} x_j\right)\right] \qquad i = 1, 2, \ldots, n$$

First Iteration:

Starting with $\left[x_1^{(0)}\ x_2^{(0)}\ x_3^{(0)}\right] = \left[0\ 0\ 0\right]$, $x_1^{(1)} = \frac{51 - 2(0) - 3(0)}{8} = 6.375$, $x_2^{(1)} = \frac{23 - 2(0) - 0}{5} = 4.6$, and $x_3^{(1)} = \frac{20 + 3(0) - 0}{6} = 3.3333$.

Second Iteration:

$$x_1^{(2)} = \frac{51 - 2(4.6) - 3(3.3333)}{8} = 3.9750,\ x_2^{(2)} = \frac{23 - 2(6.375) - 3.3333}{5} = 6.9167\text{, and}$$

$$x_3^{(1)} = \frac{20 + 3(6.375) - 4.6}{6} = 5.7542.$$

Third Iteration:

$$x_1^{(3)} = \frac{51 - 2(6.9167) - 3(5.7542)}{8} = 2.488,\ x_2^{(2)} = \frac{23 - 2(3.9750) - 5.7542}{5} = 1.8592\text{, and}$$

$$x_3^{(1)} = \frac{20 + 3(3.9750) - 6.9167}{6} = 4.1681.$$

4.16 Carry out the first three iterations of the solution of the following system of equations using the Gauss–Seidel iterative method. For the first guess of the solution, take the value of all the unknowns to be zero.

$$\begin{bmatrix} 4 & -1 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 0 \\ 1 & 0 & -1 & 4 & -1 \\ 0 & 1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

Solution

The essence of the Gauss-Seidel iterative method is given by Eq. (4.51):

$$x_i = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j=1 \\ j \neq i}}^{j=n} a_{ij}x_j\right)\right] \qquad i = 1, 2, \ldots, n$$

First Iteration:

Starting with $\left[x_1^{(0)}\ x_2^{(0)}\ x_3^{(0)}\right] = \left[0\ 0\ 0\ 0\right]$, $x_1^{(1)} = \frac{100 + 1(0) - 3(0)}{4} = 25$,

$x_2^{(1)} = \frac{100 + 1(25) + 1(0) - 1(0)}{4} = 31.25$, $x_3^{(1)} = \frac{100 + 1(31.25) + 1(0)}{4} = 32.81$,

$x_4^{(1)} = \frac{100 - 1(25) + 1(32.81) + 1(0)}{4} = 26.95$, and $x_5^{(1)} = \frac{100 - 1(31.25) + 1(26.95)}{4} = 23.93$.

Second Iteration:

$x_1^{(1)} = \frac{100 + 1(31.25) - 3(26.95)}{4} = 26.07$, $x_2^{(1)} = \frac{100 + 1(26.07) + 1(32.81) - 1(23.93)}{4} = 33.74$,

$x_3^{(1)} = \frac{100 + 1(33.74) + 1(26.95)}{4} = 40.17$, $x_4^{(1)} = \frac{100 - 1(26.07) + 1(40.17) + 1(23.93)}{4} = 34.51$, and

$x_5^{(1)} = \frac{100 - 1(33.74) + 1(34.51)}{4} = 25.19$.

Third Iteration:

$x_1^{(1)} = \frac{100 + 1(33.74) - 3(34.51)}{4} = 24.81$, $x_2^{(1)} = \frac{100 + 1(24.81) + 1(40.17) - 1(25.19)}{4} = 34.95$,

$x_3^{(1)} = \dfrac{100 + 1(34.95) + 1(34.51)}{4} = 42.36$, $x_4^{(1)} = \dfrac{100 - 1(24.81) + 1(42.36) + 1(25.19)}{4} = 35.69$, and $x_5^{(1)} = \dfrac{100 - 1(34.95) + 1(35.69)}{4} = 25.18$.

4.17 Find the condition number of the matrix in Problem 4.14 using the infinity norm.

Solution

The matrix in Problem 4.14 is $a = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$. Its inverse is $\begin{bmatrix} -0.8 & -0.6 & 0.4 \\ -0.6 & -1.2 & -0.8 \\ -0.4 & -0.8 & -1.2 \end{bmatrix}$.

The infinity norm is given by Eq.(4.73), which defines it as the maximum of the values obtained by summing the absolute values of the entries in each row.

Thus, $\|[a]\|_\infty = max\begin{bmatrix} |-2| + |1| + |0| \\ |1| + |-2| + |1| \\ |0| + |1| + |-1.5| \end{bmatrix} = max\begin{bmatrix} 3 \\ 4 \\ 2.5 \end{bmatrix} = 4$.

Similarly, $\|[a^{-1}]\|_\infty = max\begin{bmatrix} |-0.8| + |-0.6| + |0.4| \\ |-0.6| + |-1.2| + |-0.8| \\ |-0.4| + |-0.8| + |-1.2| \end{bmatrix} = max\begin{bmatrix} 1.8 \\ 2.6 \\ 2.4 \end{bmatrix} = 2.6$.

The condition number is $\|[a]\|_\infty \|[a^{-1}]\|_\infty = (4)(2.6) = 10.4$.

4.18 Find the condition number of the matrix in Problem 4.15 using the infinity norm.

Solution

The matrix in Problem 4.15 is $a = \begin{bmatrix} 8 & 2 & 3 \\ 2 & 5 & 1 \\ -3 & 1 & 6 \end{bmatrix}$. Its inverse is $\begin{bmatrix} 0.1146 & -0.0356 & -0.0514 \\ -0.0593 & 0.2253 & -0.0079 \\ 0.0672 & -0.0553 & 0.1423 \end{bmatrix}$.

The infinity norm is given by Eq.(4.73), which defines it as the maximum of the values obtained by summing the absolute values of the entries in each row.

Thus, $$\|[a]\|_\infty = max\begin{bmatrix} |8| + |2| + |3| \\ |2| + |5| + |1| \\ |-3| + |1| + |6| \end{bmatrix} = max\begin{bmatrix} 13 \\ 8 \\ 10 \end{bmatrix} = 13$$.

Similarly, $$\|[a^{-1}]\|_\infty = max\begin{bmatrix} |0.1146| + |-0.0356| + |-0.0514| \\ |-0.0593| + |0.2253| + |-0.0079| \\ |0.0672| + |-0.0553| + |0.1423| \end{bmatrix} = max\begin{bmatrix} 0.2016 \\ 0.2925 \\ 0.2648 \end{bmatrix} = 0.2925$$.

The condition number is $\|[a]\|_\infty \|[a^{-1}]\|_\infty = (13)(0.2925) = 3.8025$.

4.19 Find the condition number of the matrix in Problem 4.15 using the 1-norm.

Solution

The matrix in Problem 4.15 is $a = \begin{bmatrix} 8 & 2 & 3 \\ 2 & 5 & 1 \\ -3 & 1 & 6 \end{bmatrix}$. Its inverse is $\begin{bmatrix} 0.1146 & -0.0356 & -0.0514 \\ -0.0593 & 0.2253 & -0.0079 \\ 0.0672 & -0.0553 & 0.1423 \end{bmatrix}$.

The 1-norm is given by Eq.(4.74), which defines it as the maximum of the values obtained by summing the absolute values of the entries in each column.

Thus: $\|[a]\|_1 = max\Big[|8| + |2| + |-3| \;\; |2| + |5| + |1| \;\; |3| + |1| + |6|\Big] = max\Big[13 \;\; 8 \;\; 10\Big] = 13$

Similarly:

$$\|[a^{-1}]\|_1 = max\Big[|0.1146| + |-0.0593| + |0.0672| \;\; |-0.0356| + |0.2253| + |-0.0553| \;\; |-0.0514| + |-0.0079| + |0.1423|\Big]$$

$$= max\Big[0.2411 \;\; 0.3162 \;\; 0.2016\Big] = 0.3162$$

The condition number is: $\|[a]\|_1 \|[a^{-1}]\|_1 = (13)(0.3162) = 4.11$.

4.20 Find the condition number of the matrix in Problem 4.16 using the 1-norm.

Solution

The matrix in Problem 4.15 is $a = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 0 \\ 1 & 0 & -1 & 4 & -1 \\ 0 & 1 & 0 & -1 & 4 \end{bmatrix}$.

Its inverse is $a = \begin{bmatrix} 0.2917 & 0.0833 & 0 & -0.0833 & -0.0417 \\ 0.0833 & 0.3095 & 0.0714 & -0.0238 & -0.0833 \\ 0 & 0.0714 & 0.2857 & 0.0714 & 0 \\ -0.0833 & -0.0238 & 0.0714 & 0.3095 & 0.0833 \\ -0.0417 & -0.0833 & 0 & 0.0833 & 0.2917 \end{bmatrix}$.

The 1-norm is given by Eq.(4.74), which defines it as the maximum of the values obtained by summing the absolute values of the entries in each column.

Thus: $\|[a]\|_1 = max\begin{bmatrix} 6 & 7 & 6 & 7 & 6 \end{bmatrix} = 7$

Similarly: $\|[a^{-1}]\|_1 = max\begin{bmatrix} 0.5 & 0.5714 & 0.4286 & 0.5714 & 0.5 \end{bmatrix} = 0.5714$

The condition number is: $\|[a]\|_1 \|[a^{-1}]\|_1 = (7)(0.5714) = 4$.

4.21 Show that the eigenvalues of the $n \times n$ identity matrix are the number 1 repeated n times.

Solution

The eigenvalues of any matrix are determined from the solution of the characteristic equation, Eq.(4.93):

$$det[a - \lambda I] = 0$$

For the case of the identity matrix, $[a] = [I]$, so that Eq.(4.93) reduces to $det[I - \lambda I] = 0$, which can be recognized for an $n \times n$ matrix as:

$$det\begin{bmatrix} 1-\lambda & 0 & 0 & \dots & \dots & 0 \\ 0 & 1-\lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & 0 & 1-\lambda & 0 \\ 0 & \dots & 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^n = 0$$

which has the repeated roots $\lambda_1 = 1$, $\lambda_2 = 1$, $\dots$, $\lambda_n = 1$. Thus, the eigenvalues of the $n \times n$ identity matrix is the number 1 repeated n times.

4.22 Show that the eigenvalues of the following matrix are $1, 2, 3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution

Using Eq.(4.93), the characteristic equation for the given matrix is:

$$det[a - \lambda I] = det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 4-\lambda & 2 \\ 0 & -1 & 1-\lambda \end{bmatrix} = 0$$

Expanding the determinant yields the following characteristic equation:

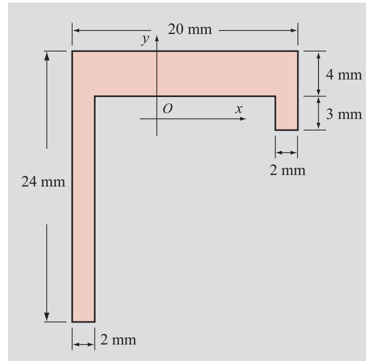
$$(1-\lambda)[(4-\lambda)(1-\lambda) - (-1)(2)] = 0$$

Clearly, $\lambda_1 = 1$ satisfies the above equation and is therefore a root. Next, $(4-\lambda)(1-\lambda) - (-1)(2) = 0$ or $\lambda^2 + 5\lambda + 6 = 0$, yielding the roots $\lambda_2 = 2$ and $\lambda_3 = 3$. Thus, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$.

4.23 The moment of inertia I_x, I_y, and the product of inertia I_{xy} of the cross-sectional area shown in the figure are:

$I_x = 5286\ \text{mm}^4$, $I_y = 4331\ \text{mm}^4$, and $I_{xy} = 2914\ \text{mm}^4$

The principal moments of inertia are the eigenvalues of the matrix $\begin{bmatrix} 5286 & 2914 \\ 2914 & 4331 \end{bmatrix}$, and the principal axes are in the direction of the eigenvectors. Determine the principal moments of inertia by solving the characteristic equation. Determine the orientation of the principal axes of inertia (unit vectors in the directions of the eigenvectors).



Solution

The eigenvalues of the matrix $\begin{bmatrix} 5286 & 2914 \\ 2914 & 4331 \end{bmatrix}$ are the principal moments of inertia. They are determined from the roots of the characteristic equation:

$$(5286 - \lambda)(4331 - \lambda) - (2914)(2914) = 0$$

which simplifies to $14402270 - 9617\lambda + \lambda^2 = 0$.

The equation is solved by using the quadratic formula:

$$\lambda_{1,2} = \frac{9617 \pm \sqrt{(9617)^2 - 4(1)(14402270)}}{2} \quad \text{or} \quad \lambda_1 = 7761.36 \quad \text{and} \quad \lambda_2 = 1855.64$$

By definition, the eigenvectors corresponding to each eigenvalue must satisfy:

$$\begin{bmatrix} 5286 & 2914 \\ 2914 & 4331 \end{bmatrix} \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \end{bmatrix} = \lambda_i \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \end{bmatrix}$$

Starting with the first eigenvalue:

$$\begin{bmatrix} 5286 & 2914 \\ 2914 & 4331 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = 7761.36 \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix}$$

These two equations are redundant and yield $u_1^{(1)} = 1.1772 u_2^{(1)}$.

Since the eigenvector is a unit vector: $(u_1^{(1)})^2 + (u_2^{(1)})^2 = 1$.

Combining the two equations yields: $u_2^{(1)} = 0.6474$ and $u_1^{(1)} = 0.7621$

Thus, the eigenvector associated with the eigenvalue $\lambda_1 = 7761.36$ is $u^{(1)} = u_1^{(1)}\hat{i} + u_2^{(1)}\hat{j} = 0.7621\hat{i} + 0.6474\hat{j}$.

Similarly, the eigenvector corresponding to the second eigenvalue is found from:

$$\begin{bmatrix} 5286 & 2914 \\ 2914 & 4331 \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = 1855.64 \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}$$

These redundant equations yield $u_2^{(2)} = (-1.177)u_1^{(2)}$. Using this result along with $(u_1^{(2)})^2 + (u_2^{(2)})^2 = 1$ yields $u_1^{(2)} = 0.6475$ and $u_2^{(2)} = -0.7621$. Thus, the eigenvector associated with the eigenvalue $\lambda_2 = 1855.64$ is $u^{(2)} = u_1^{(2)}\hat{i} + u_2^{(2)}\hat{j} = 0.6475i - 0.\hat{7}621\hat{j}$.

4.24 Determine the principal moments of inertia of the cross-sectional area in Problem 4.23 by using the QR factorization and iteration method. Carry out the first four iterations.

Solution

This is done in the MATLAB command window to preserve precision:

```
>> c=[5286;2914]; e=[1;0];
>> norm_c=sqrt((c(1)^2)+(c(2)^2)); v=c+(norm_c*e);
>> vT_v=v'*v; v_vT=v*v'; I=eye(2); H=I-((2/vT_v)*v_vT);
>> A=[5286 2914;2914 4331]; Q1=H; R1=H*A;
>> Q1*R1
ans =
  1.0e+003 *
    5.2860    2.9140
    2.9140    4.3310
>> A2=R1*Q1
A2 =
  1.0e+003 *
    7.5274   -1.1519
   -1.1519    2.0896
>> c=[7527.4;-1151.9];e=[1;0];norm_c=sqrt((c(1)^2)+(c(2)^2));
>> v=c+(norm_c*e);
>> vT_v=v'*v; v_vT=v*v'; I=eye(2); H2=I-((2/vT_v)*v_vT);
>> Q2=H2; R2=H2*A2;
>> Q2*R2
ans =
  1.0e+003 *
    7.5274   -1.1519
   -1.1519    2.0896
>> A3=R2*Q2
A3 =
  1.0e+003 *
    7.7475    0.2861
    0.2861    1.8695
>> c=[7747.5;286.1];e=[1;0];norm_c=sqrt((c(1)^2)+(c(2)^2));
>> v=c+(norm_c*e);
>> vT_v=v'*v; v_vT=v*v'; I=eye(2); H3=I-((2/vT_v)*v_vT);
>> Q3=H3; R3=H3*A3;
>> Q3*R3
ans =
  1.0e+003 *
    7.7475    0.2861
    0.2861    1.8695
>> A4=R3*Q3
```

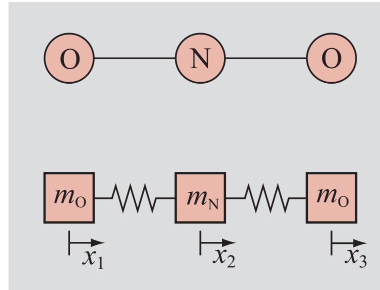
```
A4 =
  1.0e+003 *
    7.7606   -0.0686
   -0.0686    1.8564
>> c=[7760.6;-68.6]; e=[1;0]; norm_c=sqrt((c(1)^2)+(c(2)^2));
>> v=c+(norm_c*e);
>> vT_v=v'*v; v_vT=v*v'; I=eye(2); H4=I-((2/vT_v)*v_vT);
>> Q4=H4; R4=H4*A4;
>> Q4*R4
ans =
  1.0e+003 *
    7.7606   -0.0686
   -0.0686    1.8564
>> A5=R4*Q4
A5 =
  1.0e+003 *
    7.7613    0.0164
    0.0164    1.8557
```

From the above MATLAB session, it can be seen that the following sequence of matrices is generated, the last one approaching an upper triangular matrix:

$$\begin{bmatrix} 5286 & 2914 \\ 2914 & 4331 \end{bmatrix} \to \begin{bmatrix} 7527.4 & -1151.9 \\ -1151.9 & 2089.6 \end{bmatrix} \to \begin{bmatrix} 7747.5 & 286.1 \\ 286.1 & 1869.5 \end{bmatrix} \to \begin{bmatrix} 7760.6 & -68.6 \\ -68.6 & 1856.4 \end{bmatrix} \to \begin{bmatrix} 7761.3 & 16.4 \\ 16.4 & 1855.7 \end{bmatrix}$$

Thus, the approximate values of the eigenvalues are $\lambda_1 = 7761.36$ and $\lambda_2 = 1855.64$, which agrees well with the answers obtained in Problem 4.23.

4.25 The structure of an NO_2 molecule may be idealized as three masses connected by two springs, where the masses are the nitrogen and oxygen atoms and the springs represent the chemical bond between the nitrogen and oxygen atoms.The equation of motion for each atom (mass) may be written as:



$$m_O \frac{d^2 x_1}{dt^2} = -kx_1 + kx_2$$
$$m_N \frac{d^2 x_2}{dt^2} = -2kx_2 + kx_1 + kx_3$$
$$m_O \frac{d^2 x_3}{dt^2} = kx_2 - kx_3$$

where k is the restoring force spring constant representing the N–O bonds. Since the molecule is free to vibrate, normal mode (i.e., along the axis) vibrations can be examined by substituting $x_j = A_j e^{i\omega t}$, where A_j is the amplitude of the jth mass, $i = \sqrt{-1}$, ω is the frequency, and t is time. This results in the following system of equations:

$$\begin{aligned} -\omega^2 A_1 &= -\frac{k}{m_O}A_1 + \frac{k}{m_O}A_2 \\ -\omega^2 A_2 &= -\frac{2k}{m_N}A_2 + \frac{k}{m_N}A_1 + \frac{k}{m_N}A_3 \\ -\omega^2 A_3 &= \frac{k}{m_O}A_2 - \frac{k}{m_O}A_3 \end{aligned} \tag{1.1}$$

(*a*) Rewrite the system of equations in Eq. (4.145) as an eigenvalue problem, and show that the quantity ω^2 is the eigenvalue.

(*b*) Write the characteristic equation and solve (analytically) for the different frequencies.

(*c*) If $k = 14.2 \times 10^2$ kg/s^2, $m_O = 16\text{amu}$, and $m_N = 14\text{amu}$ ($1\text{amu} = 1.6605 \times 10^{-27}$kg), find the wavelengths $\lambda = \frac{2\pi c}{\omega}$ (where $c = 3 \times 10^8$ m/s is the speed of light) that correspond to the frequencies from part (*b*).

(*d*) Determine the eigenvectors corresponding to the eigenvalues found in parts (*b*) and (*c*). From the eigenvectors, deduce the relative motion of the atoms (i.e., are they moving toward, or away from each other?)

Solution

(*a*) The system of equations (4.145) can be re-written in the following matrix form:

$$\begin{bmatrix} \frac{k}{m_O} & -\frac{k}{m_O} & 0 \\ -\frac{k}{m_N} & \frac{2k}{m_N} & -\frac{k}{m_N} \\ 0 & -\frac{k}{m_O} & \frac{k}{m_O} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \omega^2 \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

It can be seen from this form that ω^2 is the eigenvalue and $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ is the eigenvector corresponding to that eigenvalue.

(b) The system in part (a) can be re-written as:

$$\begin{bmatrix} k - m_O\omega^2 & -k & 0 \\ -k & 2k - m_N\omega^2 & -k \\ 0 & -k & k - m_O\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Non-trivial solutions for the amplitudes are possible only if $det\begin{bmatrix} k - m_O\omega^2 & -k & 0 \\ -k & 2k - m_N\omega^2 & -k \\ 0 & -k & k - m_O\omega^2 \end{bmatrix} = 0$, which reduces to:

$$(k - m_O\omega^2)[(2k - m_N\omega^2)(k - m_O\omega^2) - k^2] + k[(-k)(k - m_O\omega^2)] = 0$$

or $(k - m_O\omega^2)\{(2k - m_N\omega^2)(k - m_O\omega^2) - 2k^2\} = 0$, which simplifies further to:

$$(k - m_O\omega^2)(-2km_O - m_Nk + m_Nm_O\omega^2)\omega^2 = 0$$

which yields three roots for ω : $\omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{m_O}}$, and $\omega_3 = \sqrt{\frac{k}{m_O}}\sqrt{1 + \frac{2m_O}{m_N}}$.

(c) For $k = 14.2 \times 10^2$ kg/s^2, $m_O = 16$amu and $m_N = 14$amu (amu $= 1.6605 \times 10^{-27}$ kg), the values of the frequencies are: $\omega_1 = 0$, $\omega_2 = 2.3119 \times 10^{14}\ \text{s}^{-1}$, and $\omega_3 = 4.1906 \times 10^{14}\ \text{s}^{-1}$. The latter two frequencies correspond to wavelengths of $\lambda_2 = 8.2 \times 10^{-6}$ m or $\lambda_2 = 8.2\ \mu m$ and $\lambda_3 = 4.5 \times 10^{-6}$ m or $\lambda_3 = 4.5\ \mu m$. It turns out that one of the wavelengths ($\lambda = 4.3\ \mu$m) is observed in the absorption spectrum of the NO_2 molecule.

(*d*) To find the eigenvectors, the system $\begin{bmatrix} \frac{k}{m_O} & -\frac{k}{m_O} & 0 \\ -\frac{k}{m_N} & \frac{2k}{m_N} & -\frac{k}{m_N} \\ 0 & -\frac{k}{m_O} & \frac{k}{m_O} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \omega^2 \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ must be solved for the amplitudes

for each of the eigenvalues found in part (*c*). For $\omega_1 = 0$, $\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields $A_1 = A_2 = A_3$.

This represents translation of the entire molecule, with no relative motion between the atoms. For

$\omega_2 = \sqrt{\frac{k}{m_O}}$, $\begin{bmatrix} 0 & -\frac{k}{m_O} & 0 \\ -\frac{k}{m_N} & \frac{2k}{m_N} - \frac{k}{m_O} & -\frac{k}{m_N} \\ -\frac{k}{m_O} & -\frac{k}{m_O} & \frac{k}{m_O} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which yields $A_2 = 0$ and $A_1 = -A_3$. This represents a symmetric vibration where the carbon atom is stationary and the oxygen atoms are moving in opposite directions.

For $\omega_3 = \sqrt{\frac{k}{m_O}}\sqrt{1 + \frac{2m_O}{m_C}}$, $\begin{bmatrix} -\frac{2k}{m_C} & -\frac{k}{m_O} & 0 \\ -\frac{k}{m_N} & -\frac{k}{m_O} & -\frac{k}{m_N} \\ 0 & -\frac{k}{m_O} & -\frac{2k}{m_N} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which yields $A_1 = A_3$ and $A_2 = -\frac{2m_O}{m_N}A_1$. This represents an asymmetric vibration where the carbon atom's motion is offset by the two moving oxygen atoms.

4.26 The three-dimensional state of stress at a point is given by the stress tensor:

$$\sigma_{ij} = \begin{bmatrix} 32 & -15 & 20 \\ -15 & 18 & 25 \\ 20 & 25 & 36 \end{bmatrix}$$

The principal stresses and the principal directions at the point are given by the eigenvalues and the eigenvectors. Use the power method for determining the value of the largest principal stress. Start with a column vector of 1s, and carry out the first three iterations.

Solution

The calculations are carried out in the MATLAB command window:

```
>> sigma=[32 -15 20;-15 18 25;20 25 36];
>> x1=[1;1;1]; x2=sigma*x1
x2 =
    37
    28
    81
>> x2=x2/81;x3=sigma*x2
x3 =
   29.4321
   24.3704
   53.7778
>> x3=x3/53.7778; x4=sigma*x3
x4 =
   30.7158
   24.9476
   58.2750
>> x4=x4/58.2750; x5=sigma*x4
x5 =
   30.4451
   24.7996
   57.2442
```

It can be seen that the estimate for the largest eigenvalue (i.e. largest principal stress) is 57.2442. The exact answer turns out to be 57.4296.

4.27 Modify the user-defined function `GaussPivot` in Program 4-2 (Example 4-3) such that in each step of the elimination the pivot row is switched with the row that has a pivot element with the largest absolute numerical value. For the function name and arguments use `x = GaussPivotLarge(a,b)`, where `a` is the matrix of coefficients, `b` is the right-hand-side column of constants, and `x` is the solution.
(*a*) Use the `GaussPivotLarge` function to solve the system of linear equations in Eq. (4.17).
(*b*) Use the `GaussPivotLarge` function to solve the system:

$$\begin{bmatrix} 0 & 3 & 8 & -5 & -1 & 6 \\ 3 & 12 & -4 & 8 & 5 & -2 \\ 8 & 0 & 0 & 10 & -3 & 7 \\ 3 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & -6 & 0 & 2 \\ 3 & 0 & 5 & 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 34 \\ 20 \\ 45 \\ 36 \\ 60 \\ 28 \end{bmatrix}$$

Solution

The listing of the user-defined function `GaussPivotLarge` is:

```
function x = GaussPivotLarge(a,b)
% The function solve a system of linear equations ax=b using the Gauss
% elimination method with pivoting. In each step the pivot element has the
% largest absolute numerical value.
% Input variables:
% a  The matrix of coefficients.
% b  A column vector of constants.
% Output variable:
% x  A colum vector with the solution.

ab = [a,b];
[R, C] = size(ab);
for j = 1:R-1
% Pivoting section starts.
    pvtemp=ab(j,j);
    kpvt=j;
% Looking for the row with the largest pivot element.
```

```
        for k=j+1:R
            if ab(k,j)~=0 & abs(ab(k,j)) > abs(pvtemp)
                pvtemp=ab(k,j);
                kpvt=k;
            end
        end
% If a row with a larger pivot element exists, switch the rows.
        if kpvt~=j
            abTemp=ab(j,:);
            ab(j,:)=ab(kpvt,:);
            ab(kpvt,:)=abTemp;
        end
% Pivoting section ends
    for i = j+1:R
        ab(i,j:C) = ab(i,j:C)-ab(i,j)/ab(j,j)*ab(j,j:C);
    end
end
x = zeros(R,1);
x(R) = ab(R,C)/ab(R,R);
for i = R-1:-1:1
    x(i)=(ab(i,C)-ab(i,i+1:R)*x(i+1:R))/ab(i,i);
end
```

(*a*) The following program (script file) uses the user-defined `GaussPivotLarge` function to solve the system of linear equations in Eq. (4.17).

```
% Solution of HW4_23a, script
clear all
a=[0 0.9231 0 0 0 0 0 0; -1 -0.3846 0 0 0 0 0 0
    0 0 0 0 1 0 0.8575 0; 1 0 -0.7809 0 0 0 0 0
    0 -0.3846 -0.7809 0 -1 0.3846 0 0; 0 0.9231 0.6247 0 0 -0.9231 0 0
    0 0 0.6247 -1 0 0 0 0; 0 0 0 1 0 0 -0.5145 -1];
```

```
b=[1690;3625;0;0;0;0;0;0];
x = GaussPivotLarge(a,b)
```

When the program is executed, the following result is displayed in the Command Window.

```
>> format short e
x =
 -4.3291e+003
  1.8308e+003
 -5.5438e+003
 -3.4632e+003
  2.8862e+003
 -1.9209e+003
 -3.3659e+003
 -1.7315e+003
```

(*b*) The following program (script file) uses the user-defined `GaussPivotLarge` function to solve the system of linear equations that is given in the problem statement.

```
% Solution of HW4_23b, script
clear all
a =[0 3 8 -5 -1 6; 3 12 -4 8 5 -2; 8 0 0 10 -3 7
    3 1 0 0 0 4; 0 0 4 -6 0 2; 3 0 5 0 0 -6];
b = [34; 20; 45; 36; 60; 28];
x = GaussPivotLarge(a,b)
```

When the program is executed, the following result is displayed in the Command Window.

```
>> format short
x =
  -67.1576
 -284.9310
  202.6158
  168.6108
  672.6847
  130.6010
```

4.28 Write a user-defined MATLAB function that solves a system of n linear equations, $[a][x] = [b]$, with the Gauss–Jordan method. The program should include pivoting in which the pivot row is switched with the row that has a pivot element with the largest absolute numerical value. For the function name and arguments use `x = GaussJordan(a,b)`, where `a` is the matrix of coefficients, `b` is the right-hand-side column of constants, and `x` is the solution.

(*a*) Use the `GaussJordan` function to solve the system:

$$\begin{aligned} 2x_1 + x_2 + 4x_3 - 2x_4 &= 19 \\ -3x_1 + 4x_2 + 2x_3 - x_4 &= 1 \\ 3x_1 + 5x_2 - 2x_3 + x_4 &= 8 \\ -2x_1 + 3x_2 + 2x_3 + 4x_4 &= 13 \end{aligned}$$

(*b*) Use the `GaussJordan` function to solve the system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & -3 & 2 & 5 & -4 & 6 \\ 6 & 1 & -2 & 4 & 3 & 5 \\ 3 & 2 & -1 & 4 & 5 & 6 \\ 4 & -2 & -1 & 3 & 6 & 5 \\ 5 & -6 & -3 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 91 \\ 37 \\ 63 \\ 81 \\ 69 \\ -4 \end{bmatrix}$$

Solution

The listing of the user-defined function `GaussJordan` is:

```
function x = GaussJordan(a,b)
% The function solve a system of linear equations ax=b using the Gauss
% elimination method with pivoting. In each step the rows are switched
% such that pivot element has the largest absolute numerical value.
% Input variables:
% a   The matrix of coefficients.
% b   A column vector of constants.
% Output variable:
% x   A colum vector with the solution.

ab = [a,b];
[R, C] = size(ab);
```

```
for j = 1:R
% Pivoting section starts
    pvtemp=ab(j,j);
    kpvt=j;
% Looking for the row with the largest pivot element.
    for k=j+1:R
        if ab(k,j)~=0 & abs(ab(k,j)) > abs(pvtemp)
            pvtemp=ab(k,j);
            kpvt=k;
        end
    end
% If a row with a larger pivot element exists, switch the rows.
    if kpvt~=j
        abTemp=ab(j,:);
        ab(j,:)=ab(kpvt,:);
        ab(kpvt,:)=abTemp;
    end
% Pivoting section ends
    ab(j,:)= ab(j,:)/ab(j,j);
    for i = 1:R
        if i~=j
        ab(i,j:C) = ab(i,j:C)-ab(i,j)*ab(j,j:C);
        end
    end
end
x=ab(:,C)
```

(*a*) The following program (script file) uses the user-defined `GaussJordan` function to solve the system of linear equations.

```
clear, clc
a=[2 1 4 -2; -3 4 2 -1; 3 5 -2 1; -2 3 2 4];
```

```
b=[19; 1; 8; 13];
x = GaussJordan(a,b)
```

When the program is executed, the following result is displayed in the Command Window.

```
x =
      3
      1
      4
      2
```

(*b*) The following program (script file) uses the user-defined `GaussJordan` function to solve the system of linear equations that is given in the problem statement.

```
clear, clc
a=[1 2 3 4 5 6; 1 -3 2 5 -4 6; 6 1 -2 4 3 5
    3 2 -1 4 5 6; 4 -2 -1 3 6 5; 5 -6 -3 4 -2 1];
b=[91; 37; 63; 81; 69; -4];
x = GaussJordan(a,b)
```

When the program is executed, the following result is displayed in the Command Window.

```
x =
    1.0000
    2.0000
    3.0000
    4.0000
    5.0000
    6.0000
```

4.29 Write a user-defined MATLAB function that determines the inverse of a matrix using the Gauss–Jordan method. For the function name and arguments use `Ainv = Inverse(A)`, where `A` is the matrix to be inverted, and `Ainv` is the inverse of the matrix. Use the `Inverse` function to calculate the inverse of the matrix:

$$\begin{bmatrix} -0.04 & 0.04 & 0.12 \\ 0.56 & -1.56 & 0.32 \\ -0.24 & 1.24 & -0.28 \end{bmatrix}$$

Solution

The listing of the user-defined function `Inverse` is:

```
function Ainv = Inverse(A)
% The function solve a system of linear equations ax=b using the Gauss
% elimination method.
% Input variable:
% A  The matrix to be inverted.
% Output variable:
% Ainv  the inverse of A.

[R, C] = size(A);
b=eye(R);
ab = [A,b];
CM=2*C;
for j = 1:R
    ab(j,:)= ab(j,:)/ab(j,j);
    for i = 1:R
        if i~=j
        ab(i,j:CM) = ab(i,j:CM)-ab(i,j)*ab(j,j:CM);
        end
    end
end
Ainv=ab(:,R+1:CM);
```

The function is used in the Command Window fir determining the inverse of the given function:

```
>> B = [-0.04 0.04 0.12; 0.56 -1.56 0.32; -0.24 1.24 -0.28]
B =
```

```
   -0.0400    0.0400    0.1200
    0.5600   -1.5600    0.3200
   -0.2400    1.2400   -0.2800
>> Binv = Inverse(B)
Binv =
    1.0000    4.0000    5.0000
    2.0000    1.0000    2.0000
    8.0000    1.0000    1.0000
```

To check, `B` and `Binv` are multiplied:

```
>> B*Binv
ans =
    1.0000         0   -0.0000
    0.0000    1.0000    0.0000
         0         0    1.0000
```

4.30 Write a user-defined MATLAB function that calculates the 1-norm of any matrix. For the function name and arguments use `N = OneNorm(A)`, where `A` is the matrix and `N` is the value of the norm. Use the function for calculating the 1-norm of:

(*a*) The matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$. (*b*) The matrix $B = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 0 \\ 1 & 0 & -1 & 4 & -1 \\ 0 & 1 & 0 & -1 & 4 \end{bmatrix}$.

Solution

The user-defined function is:

```
function N = OneNorm(A)
[R,C]=size(A);
for j=1:C
    Column_sums(j) = sum(abs(A(:,j)));
end
max = Column_sums(1);
for j=1:C-1
    if Column_sums(j+1) > max
        max = Column_sums(j+1);
    end
end
N = max;
```

When the user-defined function is executed in the command window for the two given matrices, the following are the results:
(*a*)

```
>> A=[-2 1 0; 1 -2 1; 0 1 -1.5];
>> N1A = OneNorm(A)
N1A =
     4
```

(*b*)

```
>> B=[4 -1 0 1 0; -1 4 -1 0 1; 0 -1 4 -1 0; 1 0 -1 4 -1; 0 1 0 -1 4];
>> N1B = OneNorm(B)
N1B =
     7
```

\4.31 Write a user-defined MATLAB function that calculates the infinity norm of any matrix. For the function name and arguments use `N = InfinityNorm(A)`, where `A` is the matrix, and `N` is the value of the norm. Use the function for calculating the infinity norm of:

(*a*) The matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1.5 \end{bmatrix}$.   (*b*) The matrix $B = \begin{bmatrix} 4 & -1 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 1 \\ 0 & -1 & 4 & -1 & 0 \\ 1 & 0 & -1 & 4 & -1 \\ 0 & 1 & 0 & -1 & 4 \end{bmatrix}$.

Solution

The user-defined function is:

```
function N = InfinityNorm(A)
[R,C]=size(A);
for i=1:R
    Row_sums(i) = sum(abs(A(i,:)));
end
max = Row_sums(1);
for i=1:R-1
    if Row_sums(i+1) > max
        max = Row_sums(i+1);
    end
end
N = max;
```

When the user-defined function is executed in the command window for the two given matrices, the following are the results:

(*a*)

```
>> A=[-2 1 0; 1 -2 1; 0 1 -1.5];
>> N1A = InfinityNorm(A)
N1A =
     4
```

(*b*)

```
>> B=[4 -1 0 1 0; -1 4 -1 0 1; 0 -1 4 -1 0; 1 0 -1 4 -1; 0 1 0 -1 4];
>> N1B = InfinityNorm(B)
N1B =
     7
```

4.32 Write a user-defined MATLAB function that calculates the condition number of an $(n \times n)$ matrix by using the 1-norm. For the function name and arguments use `c = CondNumb_One(A)`, where `A` is the matrix and `c` is the value of the condition number. Within the function, use the user-defined functions `Inverse` from Problem 4.29 and `OneNorm` from Problem 4.30. Use the function `CondNumb_One` for calculating the condition number of the matrix of the coefficients in Problem 4.29.

Solution

The listing of the user-defined function `CondNumb_One` is:

```
function c=CondNumb_One(A)
Ainv = Inverse(A);
One_norm_A = OneNorm(A);
One_norm_Ainv = OneNorm(Ainv);
c = One_norm_A*One_norm_Ainv;
```

The user-defined function is used in the Command Window for calculating the condition number of the matrix: $\begin{bmatrix} -0.04 & 0.04 & 0.12 \\ 0.56 & -1.56 & 0.32 \\ -0.24 & 1.24 & -0.28 \end{bmatrix}$

```
>> B = [-0.04 0.04 0.12; 0.56 -1.56 0.32; -0.24 1.24 -0.28]
B =
   -0.0400    0.0400    0.1200
    0.5600   -1.5600    0.3200
   -0.2400    1.2400   -0.2800
>> c=CondNumb_One(B)
c =
   31.2400
```

4.33 Write a user-defined MATLAB function that calculates the condition number of an $(n \times n)$ matrix by using the infinity norm. For the function name and arguments use `c = CondNumb_Inf(A)`, where `A` is the matrix and `c` is the value of the condition number. Within the function, use the user-defined functions `Inverse` from Problem 4.29 and `InfinityNorm` from Problem 4.31. Use the function `CondNumb_Inf` for calculating the condition number of the matrix of the coefficients in Problem 4.29.

Solution

The listing of the user-defined function `CondNumb_Inf` is:

```
function c=CondNumb_Inf(A)
Ainv = Inverse(A);
One_norm_A = InfinityNorm(A);
One_norm_Ainv = InfinityNorm(Ainv);
c = One_norm_A*One_norm_Ainv;
```

The user-defined function is used in the Command Window for calculating the condition number of the matrix: $\begin{bmatrix} -0.04 & 0.04 & 0.12 \\ 0.56 & -1.56 & 0.32 \\ -0.24 & 1.24 & -0.28 \end{bmatrix}$

```
>> B = [-0.04 0.04 0.12; 0.56 -1.56 0.32; -0.24 1.24 -0.28]
B =
   -0.0400    0.0400    0.1200
    0.5600   -1.5600    0.3200
   -0.2400    1.2400   -0.2800
>> c=CondNumb_Inf(B)
c =
   24.4000
```

4.34 Write a user-defined MATLAB function that determines the largest eigenvalue of an $(n \times n)$ matrix by using the power method. For the function name and argument use `e = MaxEig(A)`, where `A` is the matrix and `e` is the value of the largest eigenvalue. Use the function `MaxEig` for calculating the largest eigenvalue of the matrix of Problem 4.26. Check the answer by using MATLAB's built-in function for finding the eigenvalues of a matrix.

Solution

The listing of the user-defined function `MaxEig` is:

```
function e = MaxEig(A)
[R,C]=size(A);
%start with arbitrary non-zero vector transpose([1 1 1])
xx(1:R)=1;
x=xx'; xold = x;
iter = 1; reltol = 1.e-6; criterion = 1; max_old = 1;
while criterion > reltol
    xnew = A*xold;
    n = length(xnew); max=xnew(1);
    for i=1:n-1
        if abs(xnew(i+1)) > abs(max)
            max = xnew(i+1);
        end
    end
    xnew = xnew/max;
    criterion = (abs(max-max_old))/abs(max);
    xold = xnew; max_old = max;
end
e=max;
```

The user defined function `MaxEig` is used in the Command Window for calculating the largest eigenvalue of the matrix of Problem 4.26:

$$\sigma_{ij} = \begin{bmatrix} 32 & -15 & 20 \\ -15 & 18 & 25 \\ 20 & 25 & 36 \end{bmatrix}$$

```
>> Sigma=[32 -15 20; -15 18 25; 20 25 36];
>> e = MaxEig(Sigma)
e =
   57.4292
```

To check the answer, the eigenvalues of the matrix are calculated with MATLAB's built-in function `eig`:

```
>> eig(Sigma)
ans =
  -12.5262
   41.0967
   57.4296
```

The result shows that the function `MaxEig` calculated the largest eigenvalue correctly.

4.35 Write a user-defined MATLAB function that determines the smallest eigenvalue of an $(n \times n)$ matrix by using the inverse power method. For the function name and argument use `e = MinEig(A)`, where `A` is the matrix and `e` is the value of the smallest eigenvalue. Inside `MinEig` use the user-defined function `Inverse`, that was written in Problem 4.29, for calculating the inverse of the matrix `A`. Use the function `MinEig` for calculating the smallest eigenvalue of the matrix of Problem 4.26. Check the answer by using MATLAB's built-in function for finding the eigenvalues of a matrix.

Solution

The listing of the user-defined function `MinEig` is:

```
function e = MinEig(A)
Ainv = Inverse(A);
one_over_e = MaxEig(Ainv);
e = 1/one_over_e;
```

The function uses the user-defined function `Inverse` that was written in Problem 4.29, and the user-defined function `MaxEig` that was written in Problem 4.34

The user defined function `MaxEig` is used in the Command Window for calculating the largest eigenvalue of the matrix of Problem 4.26:

$$\sigma_{ij} = \begin{bmatrix} 32 & -15 & 20 \\ -15 & 18 & 25 \\ 20 & 25 & 36 \end{bmatrix}$$

```
>> Sigma=[32 -15 20; -15 18 25; 20 25 36];
>> e = MinEig(Sigma)
e =
  -12.5262
```

To check the answer, the eigenvalues of the matrix are calculated with MATLAB's built-in function `eig`:

```
>> eig(Sigma)
ans =
  -12.5262
   41.0967
```

```
57.4296
```

The result shows that the function `MaxEig` calculated the largest eigenvalue correctly.

4.36 Write a user-defined MATLAB function that determines all the eigenvalues of an $(n \times n)$ matrix by using the QR factorization and iteration method. For the function name and argument use `e=AllEig(A)`, where `A` is the matrix and `e` is a vector whose elements are the eigenvalues. Use the function `AllEig` for calculating the eigenvalues of the matrix of Problem 4.26. Check the answer by using MATLAB's built-in function for finding the eigenvalues of a matrix.

Solution

The listing of the user-defined function `AllEig` is (the user-defined function `QRFactorization` is a subfunction inside `AllEig`):

```
function e=AllEig(A)
% The function determines all eigenvalues of an (nxn) matrix
% by using the QR factorization and iteration method.
% Input variables:
% A  The (square) matrix to be factored.
% Output variables:
% e  A vector whose elements are the eigenvalues.
for i = 1:40
    [q R] = QRFactorization(A);
    A = R*q;
end
e = diag(A);

function [Q R] = QRFactorization(R)
% The function factors a matrix [A] into an orthogonal matrix [Q]
% and an upper-triangular matrix [R].
% Input variables:
% A  The (square) matrix to be factored.
% Output variables:
% Q  Orthogonal matrix.
% R  Upper-triangular matrix.

nmatrix = size(R);
```

```
n = nmatrix(1);
I = eye(n);
Q = I;
for j = 1:n-1
    c = R(:,j);
    c(1:j-1) = 0;
    e(1:n,1)=0;
    if c(j) > 0
        e(j) = 1;
    else
        e(j) = -1;
    end
    clength = sqrt(c'*c);
    v = c + clength*e;
    H = I - 2/(v'*v)*v*v';
    Q = Q*H;
    R = H*R;
end
```

The user defined function `MinEig` is used in the Command Window for calculating the smallest eigenvalue of the matrix of Problem 4.26:

$$\sigma_{ij} = \begin{bmatrix} 32 & -15 & 20 \\ -15 & 18 & 25 \\ 20 & 25 & 36 \end{bmatrix}$$

```
>> sigma=[32 -15 20;-15 18 25;20 25 36];
>> e=AllEig(sigma)
e =
   57.4296
   41.0967
  -12.5262
```

To check the answer, the eigenvalues of the matrix are calculated with MATLAB's built-in function `eig`:

```
>> d=eig(sigma)
d =
  -12.5262
   41.0967
   57.4296
```

The result shows that the function `MinEig` calculated the smallest eigenvalue correctly.