

4. Consider the function

$$f(x, y) = \sin(xy)$$

a) Write the first order Taylor expansion for this function assuming an expansion point of $(x_0, y_0) = (1, 2)$

(ans)

$$f(x_0, y_0) = \sin(x_0 y_0) = \sin(2)$$

$$\frac{\partial}{\partial x} f(x, y) = y_0 \cos(x_0 y_0) = 2 \cos(2)$$

$$\frac{\partial}{\partial y} f(x, y) = x_0 \cos(x_0 y_0) = \cos(2)$$

First order Taylor expansion is given as

$$\hat{f}_1(x, y) = f(x_0, y_0) + \frac{\partial}{\partial x} f(x, y) \Big|_{(x_0, y_0)} (x - x_0) + \frac{\partial}{\partial y} f(x, y) \Big|_{(x_0, y_0)} (y - y_0)$$

$$\hat{f}_1(x, y) = \sin(2) + 2 \cos(2)(x - x_0) + \cos(2)(y - y_0)$$

$$\hat{f}_1(x, y) = \sin(2) + 2 \cos(2)(x - 1) + \cos(2)(y - 2)$$

b) In part a what is the equation of the corresponding tangent plane?

(ans)

$$z = \sin(2) + 2 \cos(2)(x - 1) + \cos(2)(y - 2)$$

rearrange this to

$$-\sin(2) + 2 \cos(2) + 2 \cos(2) = 2 \cos(2)x + \cos(2)y - z$$

$$2 \cos(2)x + \cos(2)y - z = -\sin(2) + 4 \cos(2)$$

which is the equation of the plane in the form of $Ax + By + Cz = D$

c) What is the normal vector perpendicular to the tangent plane?

$$\hat{n} = \frac{1}{\sqrt{5 \cos^2(2) + 1}} \begin{bmatrix} 2 \cos(2) \\ \cos(2) \\ -1 \end{bmatrix}$$

5. Write an expression for the second order Taylor series of $f(x) = \ln(1+x)$ expanded about $x_0 = 0$. Plot $f(x)$ and $\hat{f}_2(x)$.

(ans) The Taylor series is given as

$$\hat{f}_2(x) = f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(0)x^2$$

where

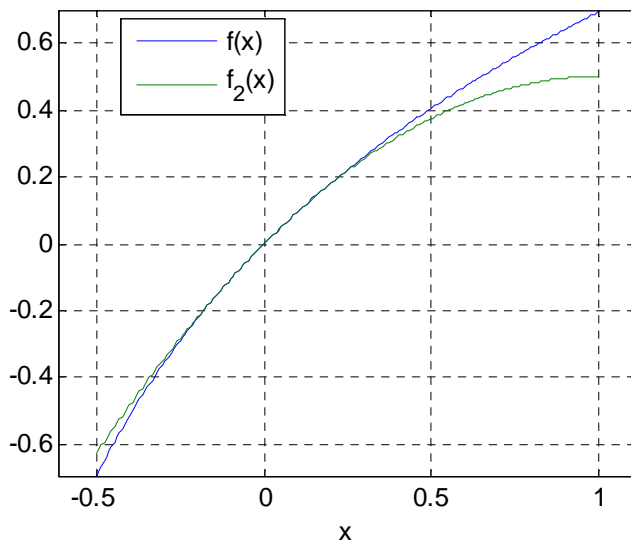
$$f(0) = \ln(1+0) = 0$$

$$f^{(1)}(0) = \left. \frac{1}{1+x} \right|_{x=0} = 1$$

$$f^{(2)}(0) = \left. \frac{-1}{(1+x)^2} \right|_{x=0} = -1$$

Such that

$$\hat{f}_2(x) = x - \frac{1}{2}x^2$$



6. Write an expression for the first order Taylor series of

$$f(x) = \ln(1+x^2)$$

expanded about $x_0 = 2$. Plot $f(x)$ and $\hat{f}_1(x)$.

(ans) The Taylor series is given as

$$\hat{f}_2(x) = f(2) + f^{(1)}(2)(x-2)$$

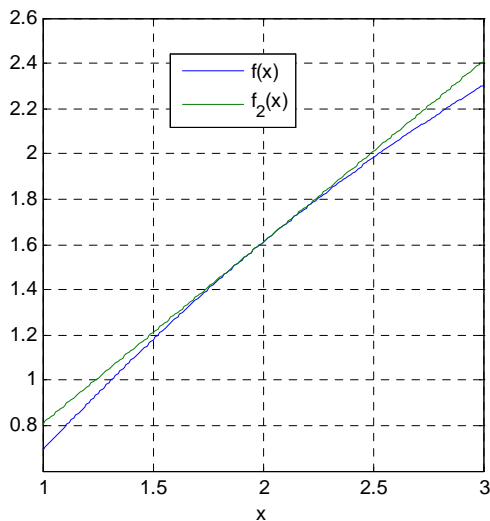
where

$$f(2) = \ln(1+2^2) = \ln(5)$$

$$f^{(1)}(2) = \left. \frac{2x}{1+x^2} \right|_{x=2} = \frac{4}{1+2^2} = \frac{4}{5}$$

Such that

$$\hat{f}_2(x) = \ln(5) + \frac{4}{5}(x-2)$$



7. Use the product rule of Calculus to prove the Taylor Series expansion. Recall that the product rule is given as

$$ab = \int adb + \int bda$$

(ans) Start with

$$\frac{d}{dt} f^{(1)}(t)(t-x) = f^{(2)}(x)(t-x) + f^{(1)}(t)$$

Now apply $\int_{x_0}^x dt$ to both sides and get

$$f^{(1)}(x_0)(x-x_0) = \int_{x_0}^x f^{(2)}(t)(t-x) dt + f(x) - f(x_0)$$

Consequently we have

$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) - \int_{x_0}^x (t-x) f^{(2)}(t) dt$$

Note that this can be written as

$$f(x) = \widehat{f}_1(x) - \int_{x_0}^x (t-x) f^{(2)}(t) dt$$

where

$$\int_{x_0}^x (t-x) f^{(2)}(t) dt$$

is the truncation error associated with $\widehat{f}_1(x)$. This truncation error term can be expanded using the product rule with

$$a(t) = f^{(2)}(t)$$

$$b(t) = \frac{(t-x)^2}{2}$$

such that

$$\begin{aligned} \int_{x_0}^x f^{(2)}(t)(t-x) dt &= \left. \frac{(t-x)^2}{2} f^{(2)}(t) \right|_{x_0}^x - \int_{x_0}^x \frac{(t-x)^2}{2} f^{(3)}(t) dt \\ &= \frac{(x-x_0)^2}{2} f^{(2)}(x_0) - \int_{x_0}^x \frac{(t-x)^2}{2} f^{(3)}(t) dt \end{aligned}$$

Putting it together we get

$$f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{(x-x_0)^2}{2} f^{(2)}(x_0) + \int_{x_0}^x \frac{(t-x)^2}{2} f^{(3)}(t) dt$$

This can be written as

$$f(x) = \widehat{f}_2(x) + \int_{x_0}^x \frac{(t-x)^2}{2} f^{(3)}(t) dt$$

where the second term is the truncation error associated with $\widehat{f}_2(x)$. Again we can expand the second integral with the product rule and eventually by induction arrive at the Taylor formula of

$$f(x) = \lim_{N \rightarrow \infty} \hat{f}_N(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x-x_0)^n}{n!}$$

8) Find the first order Taylor expansion of the function

$$f(x, y, z) = \ln(1 + xyz)$$

with $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(ans)

$$f(0, 0, 0) = \ln(1 + 0) = 0$$

$$f^{(x)}(x, y, z) = \frac{yz}{1 + xyz} = 0$$

$$f^{(y)}(x, y, z) = \frac{xz}{1 + xyz} = 0$$

$$f^{(z)}(x, y, z) = \frac{xy}{1 + xyz} = 0$$

Therefore

$$\hat{f}_1(x, y, z) = 0$$

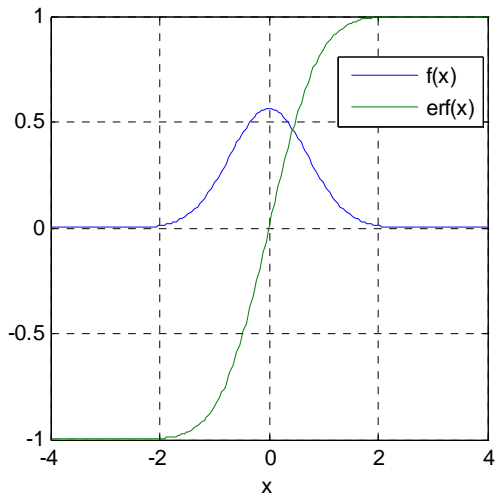
9) A special function in statistics is the error function which is defined as

$$erf(x) = \int_{-x}^x \frac{1}{\sqrt{\pi}} e^{-t^2} dt$$

Write a matlab program that will find the value of x such that $erf(x) = 0.5$ using a graphics approach, the bisection method with a bracket of [0,10]. Then use the Newton Raphson method with an initial guess of x=1. Check your answer using Matlab's `erfinv(0.5)` function. (`erfinv` is the inverse function of `erf`)

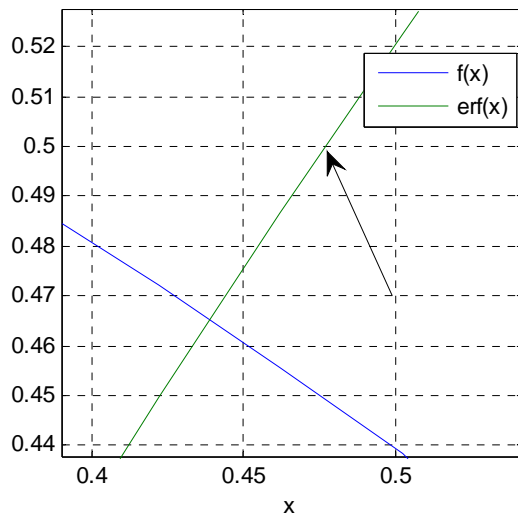
(ans)

A plot of the function $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and $erf(x)$ is given below.



```
f=@(x) 1/sqrt(pi) * exp(-x.^2);
Nx = 200;
x = linspace(-4,4,Nx)';
y = f(x);
z = erf(x);
figure(1);plot(x,[y,z]);grid on;
xlabel('x');
legend('f(x)', 'erf(x)');
```

We can zoom in on the graph for $z=0.5$ and get the following plot



This gives an approximate value of x as 0.48. Running Matlab's `erfinv(0.5)` gives x of 0.4769 which is reasonably close.

Next the bisection method is implemented by the following routine.

```
% Bisection method
g=@(x) erf(x) - 0.5;
```

```
x1 = 0;
```

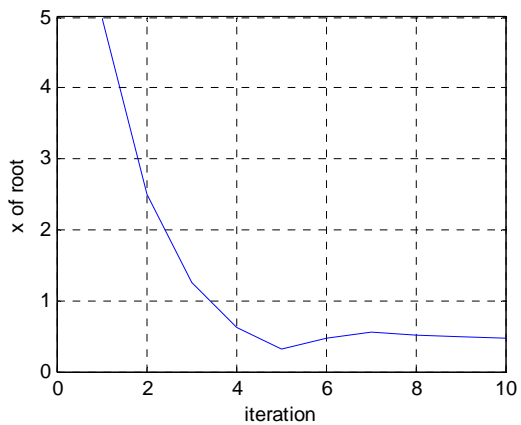
```

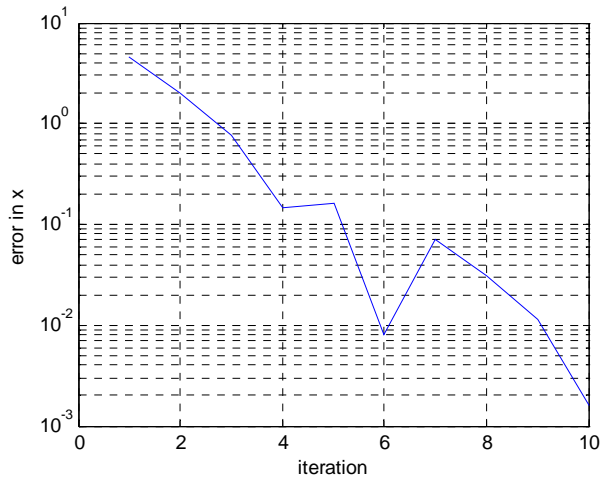
x2 = 10;
g1 = g(x1);
g2 = g(x2);
Niter = 10;
if g1*g2 >0
    error('initial bracket is not correct');
end

e = zeros(Niter,1);
xb = zeros(Niter,1);
for j=1:Niter
    xm = (x1+x2)/2;
    gm = g(xm);
    if(gm*g1)>0
        x1 = xm;
    else
        x2 = xm;
    end
    e(j) = abs(xm-erfinv(0.5));
    xb(j) = xm;
end

figure(2);
semilogy(e);grid on;
xlabel('iteration');
ylabel('error in x');
figure(3);
plot(xb);grid on;
xlabel('iteration');
ylabel('x of root');

```





Next use the Newton Raphson Method. Note that from

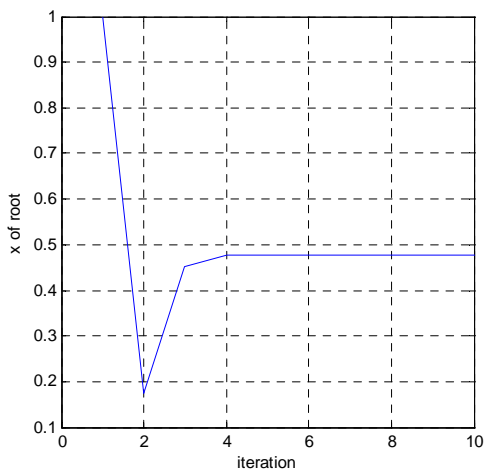
$$\operatorname{erf}(x) = \int_{-x}^x \frac{1}{\sqrt{\pi}} e^{-t^2} dt$$

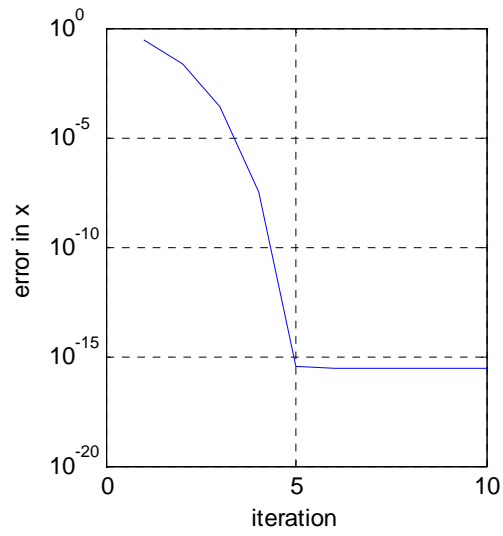
we get

$$\frac{d}{dx} \operatorname{erf}(x) = \frac{d}{dx} \int_{-x}^x \frac{1}{\sqrt{\pi}} e^{-t^2} dt = 2 \frac{1}{\sqrt{\pi}} e^{-x^2} = 2f(x)$$

Therefore the NR update is

$$x_{i+1} = x_i - \frac{\operatorname{erf}(x_i) - 0.5}{2f(x_i)}$$





Computing the answer with Matlab's built in function of `fzero()` we can write

```
>> fzero(@(x) g(x),1)
```

ans =

0.4769