MATH 3705*A
Test 4 Solutions
March 2012

[Marks] Questions 1-2 are multiple choice. Circle the correct answer. Only the answer will be marked.

[4] 1. The solution of the wave equation $u_{xx} = \frac{1}{9} u_{tt}$, $0 < x < 2$, which satisfies the boundary conditions $u(0, t) = u(2, t) = 0$, is given by

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \left[a_n \cos\left(\frac{3n\pi t}{2}\right) + b_n \sin\left(\frac{3n\pi t}{2}\right)\right].$$

If $u(x, t)$ satisfies the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = 3\sin(\pi x) - \sin(3\pi x)$, then the coefficients a_n and b_n are given by

(a) $b_2 = 3,\ b_6 = -1,\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(b) $b_2 = \frac{1}{\pi},\ b_6 = -\frac{1}{9\pi},\ b_n = 0$ otherwise, and $a_n = 0$ for all $n \geq 1$.

(c) $a_2 = -3,\ a_6 = 1,\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(d) $a_2 = -\frac{1}{\pi},\ a_6 = \frac{1}{9\pi},\ a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.

(e) None of the above

Answer: (b)

[4] 2. The solution of Laplace's equation $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ inside the circle $r = 3$ has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

The solution which satisfies the boundary condition $u(3, \theta) = 1 + 2\sin(3\theta) - 3\cos(2\theta)$ is

(a) $u(r, \theta) = 1 - 27r^{-2}\cos(2\theta) + 54r^{-3}\sin(3\theta)$

(b) $u(r, \theta) = 1 - 3r^2\cos(2\theta) + 2r^3\sin(3\theta)$

(c) $u(r, \theta) = 2 - \frac{1}{3}r^2\cos(2\theta) + \frac{2}{27}r^3\sin(3\theta)$

(d) $u(r, \theta) = 1 - \frac{1}{3}r^2\cos(2\theta) + \frac{2}{27}r^3\sin(3\theta)$

(e) None of the above

Answer: (d)

[6] 3. Find the polynomial solution $u(x,y) = \alpha x + \beta y + \gamma xy + \delta$ of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2,\ 0 < y < 1$, which satisfies the boundary conditions

$$u(0,y) = y,\ u(2,y) = 0,\ u(x,0) = 0,\ u(x,1) = 1 - \frac{1}{2}x.$$

Write down the complete solution $u(x,y)$.

Solution:

$u(x,0) = 0 \ \Rightarrow\ \alpha x + \delta = 0 \ \Rightarrow\ \alpha = \delta = 0 \ \Rightarrow\ u(x,y) = \beta y + \gamma xy.$
$u(2,y) = 0 \ \Rightarrow\ \beta y + 2\gamma y = 0 \ \Rightarrow\ (\beta + 2\gamma)y = 0 \ \Rightarrow\ \beta = -2\gamma \ \Rightarrow\ u(x,y) = -2\gamma y + \gamma xy.$
$u(0,y) = y \ \Rightarrow\ -2\gamma y = y \ \Rightarrow\ \gamma = -\frac{1}{2} \ \Rightarrow\ u(x,y) = y - \frac{1}{2}xy.$
$u(x,1) = 1 - \frac{x}{2} \ \Rightarrow\ 1 - \frac{x}{2} = 1 - \frac{x}{2}.$
Thus, $u(x,y) = y - \frac{1}{2}xy.$

[8] 4. The solution of Laplace's equation $u_{xx} + u_{yy} = 0,\ 0 < x < L,\ 0 < y < M$, satisfying the boundary conditions $u(x,0) = 0,\ u(x,M) = 0,\ u(0,y) = 0,\ u(L,y) = f(y)$, has the form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{M}\right) \sin\left(\frac{n\pi y}{M}\right).$$

Find the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangle $0 < x < 2$, $0 < y < 1$, which satisfies the boundary conditions $u(x,0) = 0,\ u(x,1) = 0,\ u(0,y) = 0$, $u(2,y) = 1$. Write down the complete solution $u(x,y)$.

Solution:

$L = 2,\ M = 1 \ \Rightarrow\ u(x,y) = \sum_{n=1}^{\infty} a_n \sinh(n\pi x)\sin(n\pi y).$

$1 = u(2,y) = \sum_{n=1}^{\infty} a_n \sinh(2n\pi)\sin(n\pi y) \ \Rightarrow$

$a_n \sinh(2n\pi) = \frac{2}{1}\int_0^1 \sin(n\pi y)\,dy = -\frac{2}{n\pi}\cos(n\pi y)\Big|_0^1 = \frac{2}{n\pi}[1 - (-1)^n] \ \Rightarrow$

$a_n = \frac{2[1-(-1)^n]}{n\pi \sinh(2n\pi)} \ \Rightarrow\ u(x,y) = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi\sinh(2n\pi)} \sinh(n\pi x)\sin(n\pi y).$

[8] 5. The solution of the wave equation $u_{xx} = \frac{1}{c^2}u_{tt},\ 0 < x < L$, which satisfies the boundary conditions $u(0,t) = u(L,t) = 0$, has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

Find the solution of $u_{xx} = u_{tt},\ 0 < x < 1$, which satisfies the boundary conditions $u(0,t) = u(1,t) = 0$, and the initial conditions $u(x,0) = 0,\ u_t(x,0) = x - x^2$. Write down the complete solution $u(x,t)$.

Solution:

Here, $c = 1$ and $L = 1$, and $u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)[a_n \cos(n\pi t) + b_n \sin(n\pi t)]$.

Since $u(x,0) = 0$, $a_n = 0,\ n \geq 1$.

$x - x^2 = u_t(x,0) = \sum_{n=1}^{\infty} n\pi b_n \sin(n\pi x) \ \Rightarrow$

$$
\begin{aligned}
n\pi b_n &= 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx \\
&= -\frac{2}{n\pi}(x - x^2)\cos(n\pi x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 (1 - 2x)\cos(n\pi x)\,dx \\
&= \frac{2}{n^2\pi^2}(1 - 2x)\sin(n\pi x)\Big|_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\,dx \\
&= -\frac{4}{n^3\pi^3}\cos(n\pi x)\Big|_0^1 = \frac{4}{n^3\pi^3}[1 - (-1)^n].
\end{aligned}
$$

Hence, $u(x,t) = \sum_{n=1}^{\infty} \frac{4[1 - (-1)^n]}{n^4\pi^4} \sin(n\pi x)\sin(n\pi t)$.