Concordia University
Department of Computer Science
COMP 238: Mathematics for Computer Science I

Sample Final Examination

1. (6%) Determine if each of the following propositions is a tautology, a contradiction, or a contingency.

   (a) $r \wedge \neg q) \wedge (q \leftrightarrow p)] \leftrightarrow \neg[(r \vee p) \rightarrow q$

   (b) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

2. (8%) Let the universe of discourse be the set of relations on the set A. Write the following English sentences as a logical expression, using quantifers, logical expressions, and set notation.

   (a) A symmetric relation on a set A is not necessarily reflexive.

   (b) Every reflexive relation is anti-symmetric.

3. (8%) Let A, B, C, and D be subsets of a universal set U.

   (a) Prove that if $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$.

   (b) Prove that

   $$(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D).$$

   (c) Let $P(X)$ denote the power set of a set X. Prove or disprove:

   $$P(A \cap B) = P(A) \cap P(B)$$

   (d) Prove or disprove:

   $$(A \times C) \cup (B \times D) \neq (A \cup B) \times (C \cup D).$$

4. (5%) Let $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$ be defined as follows, where $\mathbf{R}$ is the set of real numbers.

   $$f(x) = x^2 + 2x + 1 \quad \text{and} \quad g(x) = x^3 - 5.$$

   (a) Find f^{-1} if it exists, and if it doesn't, explain why not.

   (b) Find g^{-1} if it exists, and if it doesn't, explain why not.

5. (6%) Let $f : X \to Y$ and $g : Y \to Z$ be one-to-one correspondences.

   (a) Prove that $g \circ f$ must be a one-to-one correspondence.

   (b) Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

6. (6%) Show that if a, b, c, and m are integers such that $m \geq 2$, $c > 0$, and $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{mc}$.

7. (8%) Prove by cases that $n^2 - 1$ is divisible by 3 when n is an integer not divisible by 3.

8. (6%) Prove that $\sqrt{5}$ is irrational.

9. (6%) Use a proof by contradiction to prove the following statement: If the integers $1, 2, 3, \cdots, 7$, are placed around a circle, in any order, then there exist two adjacent integers that have a sum greater than or equal to 9.

10. (8%) Use mathematical induction to prove that $3^n + 7^n - 2$ is divisible by 8 for every non-negative integer n.

11. (6%) The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove that for every positive integer n,

$$f_3 + f_6 + \cdots + f_{3n} = \frac{1}{2}(f_{3n+2} - 1)$$

12. (6%) For each of the following relations on $\mathbf{Z}$, the set of integers, say whether it is reflexive, symmetric, anti-symmetric, or transitive.

    (a) $R = \{(a, b) \mid a^2 = b^2\}$

    (b) $R = \{(a, b) \mid |a - b| \leq 1\}$

13. (6%) For the relation $R = \{(1, 2), (1, 4), (3, 3), (4, 1)\}$ find (i) R^2 (ii) R^3 and (iii) transitive closure of R.

14. (9%) Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$.

    (a) Show that R is an equivalence relation.

    (b) What is the equivalence class of $(1, 2)$ with respect to R?

15. (6%) For the poset given by
$(\{\{2, 3\}, \{1, 3\}, \{2, 3, 5\}, \{1, 2, 3\}, \{5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 5\}\}, \subseteq)$:

    (a) Draw a Hasse diagram.

    (b) Find the maximal elements.

    (c) Find the minimal elements.

    (d) Find all upper bounds of $\{\{2, 3\}, \{5\}\}$.

    (e) Find the least upper bound of $\{\{2, 3\}, \{5\}\}$ if it exists.