MATH 3705* A Test 3 Answers and solutions March 2012

Questions 1-5 are multiple choice.

1. [3] The general solution of $xy'' + y' + 7xy = 0$ for $x > 0$ is

   (a) $c_1 J_0(\sqrt{7}\,x) + c_2 J_0(\sqrt{7}\,x)\ln(x)$ (b) $c_1 J_0(\sqrt{7}\,x) + c_2 Y_0(\sqrt{7}\,x)$

   (c) $c_1 J_{\sqrt{7}}(x) + c_2 J_{-\sqrt{7}}(x)$ (d) $c_1 J_{\sqrt{7}}(x) + c_2 Y_{\sqrt{7}}(x)$ (e) None of these

2. [3] The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ for $x > 0$ is

   (a) $c_1 J_3(\sqrt{5}\,x) + c_2 J_{-3}(\sqrt{5}\,x)$ (b) $c_1 J_3(\sqrt{5}\,x) + c_2 Y_3(\sqrt{5}\,x)$

   (c) $c_1 J_{\sqrt{5}}(3x) + c_2 J_{-\sqrt{5}}(3x)$ (d) $c_1 J_{\sqrt{5}}(3x) + c_2 Y_{\sqrt{5}}(3x)$ (e) None of these

3. [2] At $x = 20$, the Fourier Cosine series of the 8-periodic function $f(x) = \begin{cases} 3, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$ converges to

   (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) 3 (e) None of these

4. [2] The function $x^3 \sin(x) + x^2 \cos(x)$ is

   (a) even; (b) odd; (c) neither .

5. [2] If f is a periodic function of period 3, and $f(x) = 2x^2,\ 3 > x \geq 0$, then $f(97)$ is

   (a) 18; (b) 8; (c) -8; (d) 2; (e) 0.

Answers: b, b, a, a, d.

6. [8 marks] Find the Fourier cosine series of $f(x) = x - 3$ on $[0, \pi]$. Give the first three terms of the series.

Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (x-3)\,dx = \pi - 6.$$

For $n \geq 1$, $$a_n = \frac{2}{\pi}\int_0^{\pi} (x-3)\,\cos(nx)\,dx = \frac{2}{\pi}\left\{\frac{x-3}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right\}_0^{\pi} =$$

$$= \frac{2}{n^2\pi}[\cos(n\pi) - cos(0)] = \frac{2}{n^2\pi}[(-1)^n - 1].$$

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Then the series is

$$\frac{\pi-6}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\Big((-1)^n - 1\Big)\cos(nx) = \frac{\pi}{2} - 3 - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \ldots$$

7. [10 marks] The solution of the heat equation $w_{xx} = \frac{1}{\alpha^2} w_t,\ 0 < x < L$, which satisfies the boundary conditions $w(0,t) = w(L,t) = 0$, has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \frac{1}{9} u_t,\ 0 < x < 2$, which satisfies the boundary conditions $u(0,t) = 2,\ u(2,t) = 4$, and the initial condition $u(x,0) = 5$. Write down the complete solution $u(x,t)$ (give the first four terms).

Solution:

$L = 2,\ \alpha = 3$. Let $u(x,t) = v(x) + w(x,t)$, where w satisfies $w_{xx} = \frac{1}{9} w_t,\ 0 < x < 2$, $w(0,t) = w(2,t) = 0$, and $v(x)$ satisfies $v''(x) = 0,\ v(0) = 2,\ v(2) = 4$. Then $v(x) = ax + b,\ v(0) = 2 \ \Rightarrow b = 2 \ \Rightarrow v(x) = ax + 2$. Since $v(2) = 4$ then $2a+2 = 4 \ \Rightarrow a = 1 \ \Rightarrow v(x) = x+2$. Next, $w(x,0) = u(x,0) - v(x) = 5 - (x-2) = 3 - x$, so

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{9n^2\pi^2}{4} t},$$

with $3 - x = w(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \ \Rightarrow$

$$b_n = \frac{2}{2}\int_0^2 (3-x)\sin\left(\frac{n\pi x}{2}\right) dx = \left\{-\frac{2}{n\pi}(3-x)\cos\left(\frac{n\pi x}{2}\right)\right\}_0^2 - \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx =$$

$$= \frac{2}{n\pi}\left[3 - (-1)^n\right], n \geq 1.$$ Thus,

$$u(x,t) = x + 2 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[3 - (-1)^n\right] \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{9n^2\pi^2}{4} t}$$

$$= x + 2 + \frac{8}{\pi}\sin\left(\frac{\pi x}{2}\right) e^{-\frac{9\pi^2}{4} t} + \frac{2}{\pi}\sin(\pi x) e^{-9\pi^2 t} + \ldots$$

Marking Guidelines:

Problem **6**: 1 mark for a_0; 4 marks for a_n, 2 marks for the series, 1 mark for the first three terms.

Problem **7** : 1 mark for identifying L and α, 3 marks for finding $v(x)$, 3 marks for computing b_n, 2 marks for the complete solution, 1 mark for the first four terms.