CHAPTER 9

Exercise 9.2

1. (a) $f'(x) = -4x + 8 = 0$ iff $x = 2$; the stationary value $f(2) = 15$ is a relative maximum.

   (b) $f'(x) = 10x + 1 = 0$ iff $x = -1/10$; $f(-1/10) = -1/20$ is a relative minimum.

   (c) $f'(x) = 6x = 0$ iff $x = 0$; $f(0) = 3$ is a relative minimum.

2.

   (a) Setting $f'(x) = 3x^2 - 3 = 0$ yields two critical values, 1 and -1. The latter is outside the domain; the former leads to $f(1) = 3$, a relative minimum.

   (b) The only critical value is $x^* = 1$; $f(1) = 10\frac{1}{3}$ is a point of inflection.

   (c) Setting $f'(x) = -3x^2 + 9x - 6 = 0$ yields two critical values, 1 and 2; $f(1) = 3.5$ is a relative minimum but $f(2) = 4$ is a relative maximum.

3. When $x = 1$, we have $y = 2$ (a minimum); when $x = -1$, we have $y = -2$ (a maximum). These are in the nature of <u>relative</u> extrema, thus a minimum can exceed a maximum.

4. (a) $M = \phi'(x)$, $A = \phi(x)/x$

   (b) When A reaches a relative extremum, we must have

$$\frac{dA}{dx} = \frac{1}{x^2}[x\phi'(x) - \phi(x)] = 0$$

   This occurs only when $x\phi'(x) = \phi(x)$, that is, only when $\phi'(x) = \phi(x)/x$, or only when $M = A$.

   (c) The marginal and average curves must intersect when the latter reaches a peak or a trough.

   (d) $\epsilon = \frac{M}{A} = 1$ when $M = A$.

Exercise 9.3

1. (a) $f'(x) = 2ax + b$; $f''(x) = 2a$; $f'''(x) = 0$

   (b) $f'(x) = 28x^3 - 3$; $f''(x) = 84x^2$; $f'''(x) = 168x$

(c) $f'(x) = 3(1-x)^{-2}$; $f''(x) = 6(1-x)^{-3}$; $f'''(x) = 18(1-x)^{-4}$

(d) $f'(x) = 2(1-x)^{-2}$; $f''(x) = 4(1-x)^{-3}$; $f'''(x) = 12(1-x)^{-4}$

2. (a) and (b)

3. (a) An example is a modified version of the curve in Fig. 9.5a, with the arc AB replaced by a line segment AB.

   (b) A straight line.

4. Since $dy/dx = b/(c+x)^2 > 0$, and $d^2y/dx^2 = -2b/(c+x)^3 < 0$, the curve must show y increasing at a decreasing rate. The vertical intercept (where $x = 0$) is $a - \frac{b}{c}$. when x approaches infinity, y tends to the value a, which gives a horizontal asymptote. Thus the range of the function is the interval $[a - \frac{b}{c}, a)$. To use it as a consumption function, we should stipulate that:

   $a > \frac{b}{c}$ [so that consumption is positive at zero income]

   $b > c^2$ [so that $MPC = dy/dx$ is a positive fraction throughout]

5. the function $f(x)$ plots as a straight line, and $g(x)$ plots as a curve with either a peak or a bottom or an inflection point at $x = 3$. In terms of stationary points, every point on $f(x)$ is a stationary point, but the only stationary point on $g(x)$ we know of is at $x = 3$.

   (a) The utility function should have $f(0) - 0$, $f'(x) > 0$, and $f''(x) = 0$ for all x. It plots as an upward-sloping straight line emanating from the point of origin.

   (b) In the present case, the MN line segment would coincide with the utility curve. Thus points A and B lie on top of each other, and $U(15) = EU$.

Exercise 9.4

1. (a) $f'(x) = -4x+8$; $f''(x) = -4$. The critical value is $x^* = 2$; the stationary value $f(2) = 33$ is a maximum.

   (b) $f'(x) = 3x^2 + 12x$; $f''(x) = 6x + 12$. The critical values are 0 and -4. $f(0) = 9$ is a minimum, because $f''(0) = 12 > 0$, but $f''(-4) = 41$ is a maximum, because $f''(-4) = -12 < 0$.

(c) $f'(x) = x^2 - 6x + 5$; $f''(x) = 2x - 6$. The critical values are 1 and 5. $f(1) = 5\frac{1}{3}$ is a maximum because $f''(1) = -4$, but $f(5) = -5\frac{1}{3}$ is a minimum because $f''(5) = 4$.

(d) $f'(x) = 2/(1-2x)^2 \neq 0$ for any value of x; there exists no relative extremum.

2. Excluding the wall side, the other three sides must satisfy $L + 2W = 64$ft, or $L = 64 - 2W$. The area is therefore

$$A = WL = W(64 - 2W) = 64W - 2W^2$$

To maximize A, it is necessary that $dA/dW = 64 - 4W = 0$, which can occur only when $W = 16$. Thus

$$W^* = 16\text{ft} \qquad L^* = 64 - 2W^* = 32\text{ft} \qquad A^* = WL = 512\text{ft}^2$$

Inasmuch as $d^2A/dW^2 = -4$ is negative, A^* is a maximum.

3. (a) Yes.

(b) From the demand function, we first get the AR function $P = 100 - Q$. Then we have $R = PQ = (100 - Q)Q = 100Q - Q^2$.

(c) $\pi = R - C = -\frac{1}{3}Q^3 + 6Q^2 - 11Q - 50$

(d) Setting $d\pi/dQ = -Q^2 + 12Q - 11 = 0$ yields two critical values 1 and 11. Only $Q^* = 11$ gives a maximum profit.

(e) Maximum profit $= 111\frac{1}{3}$

4. If b=0, then the MC-minimizing output level becomes $Q^* = -\frac{b}{3a} = 0$. With its minimum at zero output. The MC curve must be upward-sloping throughout. Since the increasing segment of MC is associated with the convex segment of the C curve, $b = 0$ implies that the C curve will be convex throughout.

5. (a) The first assumption means $\pi(0) < 0$. Since $\pi(0) = k$, we need the restriction $k < 0$.

(b) Strict concavity means $\pi''(Q) < 0$. Since $\pi''(Q) = 2h$, we should have $h < 0$.

(c) The third assumption means $\pi'(Q^*) = 0$, or $2hQ^* + j = 0$. Since $Q^* = -j/2h$, and since $h < 0$, the positivity of Q^* requires that $j > 0$.

6. (a) $Q = f(L)$; $R = P_0Q = P_0f(L)$; $C = W_0L + F$; $\pi = R - C = P_0f(L) - W_0L - F$

(b) $d\pi/dL = P_0 f'(L) - W_0 = 0$, or $P_0 f'(L) = W_0$. The value of marginal product must be equated to the wage rate.

(c) $d^2\pi/dL^2 = P_0 f''(L)$. If $f''(L) < 0$ (diminishing MPP_L), then we can be sure that profit is maximized by L^*.

7. (a) $S = \dfrac{d}{dQ} AR = -23 + 2.2Q - 0.054Q^2$

(b) $\dfrac{dS}{dQ} = 2.2 - 0.108Q = 0$ at $Q^* = 20.37$ (approximately); since $\dfrac{d^2S}{dQ^2} = -0.108 < 0$, Q^* will maximize S.

$$S_{\max} = S|_{Q=Q^*} = -23 + 2.2(20.37) - 0.054(20.37)^2 = -0.59(\text{approximately}).$$

(c) Since $S_{\max}$ is negative, all S values must be negative.

Exercise 9.5

1. (a) 120 (b) 40320 (c) $\dfrac{4(3!)}{3!} = 4$ (d) $\dfrac{(6)(5)(4!)}{4!} = 6 \cdot 5 = 30$

(e) $\dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

2. (a)

$$\begin{aligned}
\phi(x) &= (1-x)^{-1} & \text{so that} \quad \phi(0) &= 1\\
\phi'(x) &= (1-x)^{-2} & \phi'(0) &= 1\\
\phi''(x) &= 2(1-x)^{-3} & \phi''(0) &= 2\\
\phi'''(x) &= 6(1-x)^{-4} & \phi'''(0) &= 6\\
\phi^{(4)}(x) &= 24(1-x)^{-5} & \phi^{(4)}(0) &= 24
\end{aligned}$$

Thus, according to (9.14), the first five terms are $1 + x + x^2 + x^3 + x^4$

(b)

$$\begin{aligned}
\phi(x) &= (1-x)/(1+x) & \text{so that} \quad \phi(0) &= 1\\
\phi'(x) &= -2(1+x)^{-2} & \phi'(0) &= -2\\
\phi''(x) &= 4(1+x)^{-3} & \phi''(0) &= 4\\
\phi'''(x) &= -12(1+x)^{-4} & \phi'''(0) &= -12\\
\phi^{(4)}(x) &= 48(1+x)^{-5} & \phi^{(4)}(0) &= 48
\end{aligned}$$

Thus, by (9.14), the first five terms are $1 - 2x + 2x^2 - 2x^3 + 2x^4$

3. (a) $\phi(-2) = 1/3$, $\phi'(-2) = 1/9$, $\phi''(-2) = 2/27$, $\phi'''(-2) = 6/81$, and $\phi^{(4)}(-2) = 24/243$. Thus, by (9.14),

$$\begin{aligned}\phi(x) &= \frac{1}{3} + \frac{1}{9}(x+2) + \frac{1}{27}(x+2)^2 + \frac{1}{81}(x+2)^3 + \frac{1}{243}(x+2)^4 + R_4 \\ &= \frac{1}{243}(211 + 131x + 51x^2 + 11x^3 + x^4) + R_4\end{aligned}$$

(b) $\phi(-2) = -3$, $\phi'(-2) = -2$, $\phi''(-2) = -4$, $\phi'''(-2) = -12$, and $\phi^{(4)}(-2) = -48$. Thus, by (9.14),

$$\begin{aligned}\phi(x) &= -3 - 2(x+2) - 2(x+2)^2 - 2(x+2)^3 - 2(x+2)^4 + R_4 \\ &= -63 - 98x - 62x^2 - 18x^3 - 2x^4 + R_4\end{aligned}$$

4. When $x = x_0$, all the terms on the right of (9.14) except the first one will drop out (including R_n), leaving the result $\phi(x) = \phi(x_0)$.

Exercise 9.6

1. (a) $f'(x) = 3x^2 = 0$ only when $x = 0$, thus $f(0) = 0$ is the only stationary value. The first nonzero derivative value is $f'''(0) = 6$; so $f(0)$ is an inflection point.

   (b) $f'(x) = -4x^3 = 0$ only when $x = 0$. The stationary value $f(0) = 0$ is a relative maximum because the first nonzero derivative value is $f^{(4)}(0) = -24$.

   (c) $f'(x) = 6x^5 = 0$ only when $x = 0$. The stationary value $f(0) = 5$ is a relative minimum since the first nonzero derivative value is $f^{(6)}(0) = 720$.

2. (a) $f'(x) = 3(x-1)^2 = 0$ only when $x = 1$. The first nonzero derivative value is $f'''(1) = 6$. Thus the stationary value $f(1) = 16$ is associated with an inflection point.

   (b) $f'(x) = 4(x-2)^3 = 0$ only when $x = 2$. Since the first nonzero derivative value is $f^{(4)}(2) = 24$, the stationary value $f(2) = 0$ is a relative minimum.

   (c) $f'(x) = -6(3-x)^5 = 0$ only when $x = 3$. Since the first nonzero derivative value is $f^{(6)}(3) = 720$, the stationary value $f(3) = 7$ is a relative minimum.

   (d) $f'x) = -8(5-2x)^3 = 0$ only when $x = 2.5$,. Since the first nonzero derivative value is $f^{(4)}(2.5) = 384$, the stationary value $f(2.5) = 8$ is a relative minimum.

CHAPTER 11

Exercise 11.2

1. The derivatives are: $f_x = 2x + y$, $f_y = x + 4y$, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 1$. The first-order condition requires that $2x + y = 0$ and $x + 4y = 0$. Thus we have

   $$x^* = y^* = 0 \qquad \text{implying} \qquad z^* = 3 \qquad \text{(which is a minimum)}$$

2. The derivatives are: $f_x = -2x + 6$, $f_y = -2y + 2$, $f_{xx} = -2$, $f_{yy} = -2$, and $f_{xy} = 0$. The first-order condition requires that $-2x = 6$ and $-2y = -2$. Thus we find

   $$x^* = 3 \qquad y^* = 1 \qquad \text{so that} \qquad z^* = 10 \ \ \text{(which is a maximum)}$$

3. $f_x = 2ax$, $f_y = 2by$, $f_{xx} = 2a$, $f_{yy} = 2b$, and $f_{xy} = 0$. The first-order condition requires that $2ax = 0$ and $2by = 0$. Thus

   $$x^* = y^* = 0 \qquad \text{so that} \qquad z^* = c$$

   The second derivatives give us $f_{xx}f_{yy} = 4ab$, and $f_{xy}^2 = 0$. Thus:

   (a) z^* is a minimum if a, $b > 0$.

   (b) z^* is a maximum if a, $b < 0$.

   (c) z^* gives a saddle point if a and b have opposite signs.

4. $f_x = 2\left(e^{2x} - 1\right)$, $f_y = 4y$, $f_{xx} = 4e^{2x}$, $f_{yy} = 4$, and $f_{xy} = 0$. The first-order condition requires that $e^{2x} = 1$ and $4y = 0$. Thus

   $$x^* = y^* = 0 \qquad \text{so that} \qquad z^* = 4$$

   Since $f_{xx}f_{yy} = 4(4)$ exceeds $f_{xy}^2 = 0$, $z^* = 4$ is a minimum.

5.

   (a) And pair (x, y) other than $(2, 3)$ yields a positive z value.

   (b) Yes. At $x^* = 2$ and $y^* = 3$, we find

   $$f_x = 4\left(x - 2\right)^3 \qquad \text{and} \qquad f_y = 4\left(y - 3\right)^3 = 0$$

   (c) No. At $x^* = 2$ and $y^* = 3$, we have $f_{xx} = f_{yy} = f_{xy} = f_{yx} = 0$.

   (d) By (11.6), $d^2z = 0$. Thus (11.9) is satisfied.

Exercise 11.3

1. (a) $q = 4u^2 + 4uv + 3v^2$

   (b) $q = -2u^2 + 4uv - 4v^2$

   (c) $q = 5x^2 + 6xy$

   (d) $q = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}dy^2$

2. For (b): $q = -2u^2 + 4uv - 4v^2$. For (c): $q = 5x^2 + 6xy$. Both are the same as before.

3.

   (a) $\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$: $4 > 0$, $4(3) > 2^2$ – positive definite

   (b) $\begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$: $-2 < 0$, $-2(-4) > 2^2$ – negative definite

   (c) $\begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$: $5 > 0$, $5(0) < 3^2$ – neither

4.

   (a) $q = [u \;\; v]\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

   (b) $q = [u \;\; v]\begin{bmatrix} 1 & 3.5 \\ 3.5 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

   (c) $q = [u \;\; v]\begin{bmatrix} -1 & 4 \\ 4 & -31 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

   (d) $q = [x \;\; y]\begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

   (e) $q = [u_1 \;\; u_2 \;\; u_3]\begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

   (f) $q = [u \;\; v \;\; w]\begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -7 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

5.

(a) $3 > 0$, $3(7) > (-2)^2$ – positive definite

(b) $1 > 0$, $1(3) < (3.5)^2$ – neither

(c) $-1 < 0$, $-1(-31) > 4^2$ – negative definite

(d) $-2 < 0$, $-2(-5) > 3^2$ – negative definite

(e) $3 > 0$, $\begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} = 14 > 0$, $\begin{vmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{vmatrix} = 37 > 0$ – positive definite

(f) $-1 < 0$, $\begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0$ – neither (no need to check $|D_3|$)

6.

(a) The characteristic equation is

$$\begin{vmatrix} 4-r & 2 \\ 2 & 3-r \end{vmatrix} = r^2 - 7r + 8 = 0$$

Its roots are $r_1, r_2 = \frac{1}{2}\left(7 + \sqrt{17}\right)$. Both roots being positive, $u'Du$ is positive definite.

(b) The characteristic equation is $r^2 + 6r + 4 = 0$, with roots $r_1, r_2 = -3 \pm \sqrt{5}$. Both roots being negative, $u'Eu$ is negative definite.

(c) The characteristic equation is $r^2 - 5r - 9 = 0$, with roots $r_1, r_2 = \frac{1}{2}\left(5 \pm \sqrt{61}\right)$. Since r_1 is positive, but r_2 is negative, $u'Fu$ is indefinite.

7. The characteristic equation $\begin{vmatrix} 4-r & 2 \\ 2 & 1-r \end{vmatrix} = r^2 - 5r = 0$ has the roots $r_1 = 5$ and $r_2 = 0$. (Note: This is an example where $|D| = 0$). Using r_1 in (11.13'), we have $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Thus $x_1 = 2x_2$. Upon normalization, we obtain the first characteristic vector

$$v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

Next, using r_2 in (11.13'), we have $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Therefore, $x_1 = -\frac{1}{2}x_2$. Upon normalization, we obtain

$$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

These results happen to be identical with those in Example 5.

8. The characteristic equation can be written as

$$r^2 - (d_{11} + d_{22})\,r + (d_{11}d_{22} - d_{12}d_{21}) = 0$$

Thus $r_1,\, r_2 = \frac{1}{2}\left[(d_{11} + d_{22}) \pm \sqrt{(d_{11} + d_{22})^2 - 4\,(d_{11}d_{22} - d_{12}d_{21})}\right]$

(a) The expression under the square-root sign can be written as

$$\begin{aligned} E &= d_{11}^2 + 2d_{11}d_{22} + d_{22}^2 - 4d_{11}d_{22} + 4d_{12}d_{21} \\ &= d_{11}^2 - 2d_{11}d_{22} + d_{22}^2 + 4d_{12}d_{21} = (d_{11} - d_{22})^2 + 4d_{12}^2 \geq 0 \end{aligned}$$

Thus no imaginary number can occur in r_1 and r_2.

(b) To have repeated roots, E has to be zero, which can occur if and only if $d_{11} = d_{22}$ (say, =c) and at the same time $d_{12} = d_{21} = 0$. This would mean that matrix D takes the form of $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$.

(c) Positive or negative semidefiniteness allows a characteristic root to be zero (r=0), which implies the possibility that the characteristic equation reduces to $d_{11}d_{22} - d_{12}d_{21} = 0$, or $|D| = 0$.

Exercise 11.4

1. The first-order condition

$$f_1 = 2x_1 - 3x_2 = 0$$

$$f_2 = -3x_1 + 6x_2 + 4x_3 = 0$$

$$f_3 = 4x_2 + 12x_3 = 0$$

is a homogeneous linear-equation system in which the three equations are independent. Thus the only solution is

$$x_1^* = x_2^* = x_3^* = 0 \qquad \text{so that} \qquad z^* = 0$$

The Hessian is $\begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$, with $|H_1| = 2 > 0$, $|H_2| = 3 > 0$, and $|H_3| = 4 > 0$. Consequently, $z^* = 0$ is a minimum.

2. The first-order condition consists of the three equations

   $$f_1 = -2x_1 = 0 \qquad f_2 = -2x_2 = 0 \qquad f_3 = -2x_3 = 0$$

   Thus $\quad x_1^* = x_2^* = x_3^* = 0 \quad$ so that $\quad z^* = 29$

   The Hessian is $\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$, with $|H_1| = -2 < 0$, $|H_2| = 4 > 0$, and $|H_3| = -8 < 0$.
   Consequently, $z^* = 29$ is a maximum.

3. The three equations in the first-order conditions are

   $$2x_1 + x_3 = 0$$

   $$2x_2 + x_3 = 1$$

   $$x_1 + x_2 + 6x_3 = 0$$

   Thus $\quad x_1^* = \frac{1}{20} \quad x_2^* = \frac{11}{20} \quad x_3^* = -\frac{2}{20} \quad$ so that $\quad z^* = -\frac{11}{40}$

   Since the Hessian is $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}$, with $|H_1| = 2 > 0$, $|H_2| = 4 > 0$, and $|H_3| = 20 > 0$, the z^* value is a minimum.

4. By the first-order condition, we have

   $$f_x = 2e^{2x} - 2 = 0, \qquad f_y = -e^{-y} + 1 = 0, \qquad f_w = 2we^{w^2} - 2e^w = 0$$

   Thus $\quad x^* = 0 \quad y^* = 0 \quad \bar{w} = 1 \quad$ so that $\quad z^* = 2 - e$

   Note:The values of x^* and y^* are found from the fact that $e^0 = 1$. Finding w^* is more complicated. One way of doing it is as follows: First, rewrite the equation $f_w = 0$ as

   $$we^{w^2} = e^w$$

   Taking natural logs yield

   or

   $$\ln w + \ln e^{w^2} = \ln e^w$$
   $$\text{or} \quad \ln w + w^2 = w$$
   $$\text{or} \quad \ln w = w - w^2$$

If we draw a curve for $\ln w$, and another for $w - w^2$, their intersection point will give us the solution. The $\ln w$ curve is a strictly concave curve with horizontal intercept at $w = 1$. The $w - w^2$ is a hill-type parabola with horizontal intercepts $w = 0$ and $w = 1$. Thus the solution is $w^* = 1$.

The Hessian is $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4e \end{vmatrix}$ when evaluated at the stationary point, with all leading principal minors positive. Thus z^* is a minimum.

5.

(a) Problems 2 and 4 yield diagonal Hessian matrices. The diagonal elements are all negative for problem 2, and all positive for problems 4 and 5.

(b) According to (11.16), these diagonal elements represent the characteristic roots. Thus the characteristic roots are all negative (d^2z negative definite) for problem 2, and all positive (d^2z positive definite) for problem 4.

(c) Yes.

6.

(a) The characteristic equation is, by (11.14):

$$\begin{vmatrix} 2-r & 0 & 1 \\ 0 & 2-r & 1 \\ 1 & 1 & 6-r \end{vmatrix} = 0$$

Expanding the determinant by the method of Fig. 5.1, we get

$$(2-r)(2-r)(6-r) - (2-r) - (2-r) = 0$$

or $(2-r)[(2-r)(6-r) - 2] = 0$ [factoring]

or $(2-r)\left(r^2 - 8r + 10\right) = 0$

Thus, from the $(2-r)$ term, we have $r_1 = 2$. By the quadratic formula, we get from the other term: $r_2, r_3 = 4 \pm \sqrt{6}$.

(b) All three roots are positive. Thus d^2z is positive definite, and z^* is a minimum.

(c) Yes.

Exercise 11.5

1.

(a) Let u and v be any two distinct points in the domain. Then

$$f(u) = u^2 \qquad f(v) = v^2 \qquad f[\theta u + (1-\theta)v] = [\theta u + (1-\theta)v]^2$$

Substituting these into (11.20), we find the difference between the left- and right-side expressions in (11.20) to be

$$\theta u^2 + (1-\theta)v^2 - \theta^2 u^2 - 2\theta(1-\theta)uv - (1-\theta)^2 v^2$$
$$= (1-\theta)u^2 - 2\theta(1-\theta)uv + \theta(1-\theta)v^2$$
$$= (1-\theta)(u-v)^2 > 0 \qquad [\text{since } u \neq v\ ]$$

Thus $z = x^2$ is a strictly convex function.

(b) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two distinct points in the domain. Then

$$f(u) = u_1^2 + 2u_2^2 \qquad f(v) = v_1^2 + 2v_2^2$$
$$f[\theta u + (1-\theta)v] = [\theta u_1 + (1-\theta)v_1]^2 + 2[\theta u_2 + (1-\theta)v_2]^2$$

The difference between the left- and right-side expressions in (11.20) is

$$\theta(1-\theta)\left(u_1^2 - 2u_1v_1 + v_1^2 + 2u_2^2 - 4u_2v_2 + 2v_2^2\right) = \theta(1-\theta)\left[(u_1-v_1)^2 + 2(u_2-v_2)^2\right] > 0$$

Thus $z = x_1^2 + 2x_2^2$ is a strictly convex function.

(c) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two distinct points in the domain. Then

$$\begin{aligned} f(u) &= 2u_1^2 - u_1u_2 + u_2 \qquad fv = 2v_1^2 - v_1v_2 + v_2^2 \\ f[\theta u + (1-\theta)v] &= 2\left[\theta u_1 + (1-\theta)v_1^2\right] - [\theta u_1 + (1-\theta)v_1]\cdot[\theta u_2 + (1-\theta)v_2] \\ &\quad + [\theta u_2 + (1-\theta)v_2]^2 \end{aligned}$$

The difference between the left- and right-side expressions in (11.20) is

$$\theta(1-\theta)\left[\left(2u_1^2 - 4u_1v_1 + 2v_1^2\right) - u_1u_2 + u_1v_2 + v_1u_2 - v_1v_2 + \left(u_2^2 - 2u_2v_2 + v_2^2\right)\right]$$
$$= \theta(1-\theta)\left[2(u_1-v_1)^2 - (u_1-v_1)(u_2-v_2) + (u_2-v_2)^2\right] > 0$$

because the bracketed expression is positive, like $\theta(1-\theta)$. [The bracketed expression, a positive-definite quadratic form in the two variables $(u_1 - v_1)$ and $(u_2 - v_2)$, is positive since $(u_1 - v_1)$ and $(u_2 - v_2)$ are not both zero in our problem.] Thus $z = 2x^2 - xy + y^2$ is a strictly convex function.

2.

(a) With $f'(u) = -2u$, the difference between the left- and right-side expressions in (11.24) is

$$-v^2 + u^2 + 2u(v-u) = -v^2 + 2uv - u^2 = -(v-u)^2 < 0$$

Thus $z = -x^2$ is strictly concave.

(b) Since $f_1(u_1,u_2) = f_2(u_1,u_2) = 2(u_1+u_2)$, the difference between the left- and right-side expressions in (11.24') is

$$(v_1+v_2)^2 - (u_1+u_2)^2 - 2(u_1+u_2)[(v_1-u_1)+(v_2-u_2)]$$
$$= (v_1+v_2)^2 - 2(v_1+v_2)(u_1+u_2) + (u_1+u_2)^2$$
$$= [(v_1+v_2)-(u_1+u_2)]^2 \geq 0$$

A zero value cannot be ruled out because the two points may be, e.g., $(u_1,u_2) = (5,3)$ and $(v_1,v_2) = (2,6)$. Thus $z = (x_1+x_2)^2$ is convex, but not strictly so.

(c) Since $f_1(u_1,u_2) = -u_2$, and $f_2(u_1,u_2) = -u_1$, the difference between the left- and right-side expressions in (11.24') is

$$-v_1v_2 + u_1u_2 + u_2(v_1-u_1) + u_1(v_2-u_2) = -v_1v_2 + v_1u_2 + u_1v_2 - u_1u_2 = (v_1-u_1)(v_2-u_2) \gtreqless 0$$

Thus $z = -xy$ is neither convex nor concave.

3. No. That theorem gives a sufficient condition which is not satisfied.

4.

(a) No.

(b) No.

(c) Yes.

5.

(a) The circle with its interior, i.e. a disk.

(b) Yes.

6.

(a) The set of points on an exponential curve; not a convex set.

(b) The set of points lying on or above an exponential curve; a convex set.

(c) The set of points lying on or below an inverse U-shaped curve; a convex set.

(d) The set of points lying on or above a rectangular hyperbola in the positive quadrant; a convex set.

7.

(a) This is a convex combination, with $\theta = 0.5$.

(b) This is again a convex combination, with $\theta = 0.2$.

(c) This is not a convex combination.

8.

(a) This set is the entire 2-space.

(b) This set is a cone bounded on one side by a ray passing through point u, and on the other side by a ray passing through point v.

(c) This set is the line segment uv.

9.

(a) $S^{\leq} \equiv \{(x_1, ..., x_n) \mid f(x_1, ..., x_n) \leq k\}$ (f convex)
$S^{\geq} \equiv \{(x_1, ..., x_n) \mid g(x_1, ..., x_n) \geq k\}$ (g concave)

(b) $S^{\leq}$ is a solid circle (or disk); $S^{\geq}$ is a solid square.

Exercise 11.6

1.

(a) No, because the marginal cost of one commodity will be independent of the output of the other.

(b) The first-order condition is

$$\pi_1 = P_{10} - 4Q_1 = 0 \qquad \pi_2 = P_{20} - 4Q_2 = 0$$

Thus $Q_1^* = \frac{1}{4}P_{10}$ and $Q_2^* = \frac{1}{4}P_{20}$. The profit is maximized, because the Hessian is $\begin{vmatrix} -4 & 0 \\ 0 & -4 \end{vmatrix}$, with $|H_1| < 0$ and $|H_2| > 0$. The signs of the principal minors do not depend on where they are evaluated. Thus the maximum in this problem is a unique absolute maximum.

(c) $\pi_{12} = 0$ implies that the profit-maximizing output level of one commodity is independent of the output of the other (see first-order condition). The firm can operate as if it has two plants, each optimizing the output of a different product.

2.

(a) By the procedure used in Example 2 (taking Q_1 and Q_2 as choice variables), we can find

$$Q_1^* = 3\tfrac{4}{7} \qquad Q_2^* = 4\tfrac{9}{14} \qquad P_1^* = 6\tfrac{1}{14} \qquad P_2^* = 24\tfrac{2}{7}$$

(b) The Hessian is $\begin{vmatrix} -4 & 2 \\ 2 & -8 \end{vmatrix}$, with $|H_1| = -4$ and $|H_2| = 28$. Thus the sufficient condition for a maximum is met.

(c) Substituting the P^*'s and Q^*'s into the R and C functions, we get

$$R^* = 134\tfrac{43}{98} \qquad C^* = 65\tfrac{85}{98} \qquad \text{and} \qquad \bar{r} = 68\tfrac{4}{7}$$

3. $|c_{d1}| = \left|\frac{dQ_1}{dP_1}\frac{P_1^*}{Q_1^*}\right| = \frac{1}{4}\frac{39}{6} = \frac{13}{8}$. Similarly, $|c_{d2}| = \frac{1}{5}\frac{60}{9} = \frac{4}{3}$, and $|c_{d3}| = \frac{1}{6}\frac{45}{5} = \frac{3}{2}$. The highest is $|c_{d1}|$; the lowest is $|c_{d2}|$.

(a) $C' = 15 + 2Q = 15 + 2Q_1 + 2Q_2 + 2Q_3$

(b) Equating each MR to the MC, we obtain the three equations:

$$10Q_1 + 2Q_2 + 2Q_3 = 48, \qquad 2Q_1 + 12Q_2 + 2Q_3 = 90 \qquad \text{and} \qquad 2Q_1 + 2Q_2 + 14Q_3 = 60$$

Thus $Q_1^* = 2\frac{88}{97}$, $Q_2^* = 6\frac{51}{97}$, $Q_3^* = 2\frac{91}{97}$.

(c) Substituting the above into the demand equations, we get

$$P_1^* = 51\frac{36}{97}, \qquad P_2^* = 72\frac{36}{97}, \qquad P_3^* = 57\frac{36}{97}$$

(d) Since $R_1'' = -8$, $R_2'' = -10$, $R_3'' = -12$, and $C'' = 2$, we do find that:
(1) $R_1'' - C'' = -10$ (2) $R_1''R_2'' - (R_1'' + R_2'')\,C'' = 80 + 36 = 116 > 0$, and (3) $|H| = -960 - (80 + 96 + 120)\,(2) = -1552 < 0$.

4.

(a) $\pi = P_0 Q(a,b)\left(1 + \frac{1}{2}i_0\right)^{-2} - P_{a0}a - P_{b0}b$

(b) $\pi = P_0 Q(a,b)\left(1 + \frac{1}{4}i_0\right)^{-3} - P_{a0}a - P_{b0}b$

5. $Q(a,b) = 260$

Exercise 11.7

1.

(a) We may take (11.49) as the point of departure. Letting P_{a0} alone vary (i.e., letting $dP_0 = dP_{b0} = dr = dt = 0$), and dividing through by $dP_{a0} \neq 0$, we get the matrix equation

$$\begin{bmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial a^*}{\partial P_{a0}}\right) \\ \left(\frac{\partial b^*}{\partial P_{a0}}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, by Cramer's Rule,

$$\left(\frac{\partial a^*}{\partial P_{a0}}\right) = \frac{P_0 Q_{bb} e^{-rt}}{|J|} < 0 \qquad \text{and} \qquad \left(\frac{\partial b^*}{\partial P_{a0}}\right) = -\frac{P_0 Q_{ab} e^{-rt}}{|J|} < 0$$

The higher the price of input a, the smaller will be the equilibrium levels of inputs a and b.

(b) Next, letting P_{b0} alone vary in (11.49), and dividing through by $dP_{b0} \neq 0$, we can obtain results similar to (a) above:

$$\left(\frac{\partial a^*}{\partial P_{b0}}\right) = -\frac{P_0 Q_{ab} e^{-rt}}{|J|} < 0 \qquad \text{and} \qquad \left(\frac{\partial b^*}{\partial P_{b0}}\right) = \frac{P_0 Q_{aa} e^{-rt}}{|J|} < 0$$

2.

(a) P_0, i_0, P_{a0}, P_{b0}.

CHAPTER 12

Exercise 12.2

1.

(a) $Z = xy + \lambda(2 - x - 2y)$. The necessary condition is:

$$Z_\lambda = 2 - x - 2y = 0 \qquad Z_x = y - \lambda = 0 \qquad Z_y = x - 2\lambda = 0$$

Thus $\lambda^* = \frac{1}{2}$, $x^* = 1$, $y^* = \frac{1}{2}$ – yielding $z^* = \frac{1}{2}$.

(b) $Z = xy + 4x + \lambda(8 - x - y)$. The necessary condition is:

$$Z_\lambda = 8 - x - y = 0 \qquad Z_x = y + 4 - \lambda = 0 \qquad Z_y = x - \lambda = 0$$

Thus $\lambda^* = 6$, $x^* = 6$, $y^* = 2$ – yielding $z^* = 36$.

(c) $Z = x - 3y - xy + \lambda(6 - x - y)$. The necessary condition is:

$$Z_\lambda = 6 - x - y = 0 \qquad Z_x = 1 - y - \lambda = 0 \qquad Z_y = -3 - x - \lambda = 0$$

Thus $\lambda^* = -4$, $x^* = 1$, $y^* = 5$ – yielding $z^* = -19$.

(d) $Z = 7 - y + x^2 + \lambda(-x - y)$. The necessary condition is:

$$Z_\lambda = -x - y = 0 \qquad Z_x = 2x - \lambda = 0 \qquad Z_y = -1 - \lambda = 0$$

Thus $\lambda^* = -1$, $x^* = -\frac{1}{2}$, $y^* = \frac{1}{2}$ – yielding $z^* = 6\frac{3}{4}$.

2.

(a) Increase; at the rate $\frac{dz^*}{dc} = \lambda^* = \frac{1}{2}$.

(b) Increase; $\frac{dz^*}{dc} = 6$.

(c) Decrease; $\frac{dz^*}{dc} = -4$

(d) Decrease; $\frac{dz^*}{dc} = -1$

3.

(a) $Z = x + 2y + 3w + xy - yw + \lambda(10 - x - y - 2w)$. Hence:

$$Z_\lambda = 10 - x - y - 2w = 0 \qquad Z_x = 1 + y - \lambda = 0$$

$$Z_y = 2 + x - w - \lambda = 0 \qquad Z_w = 3 - y - 2\lambda = 0$$

(b) $Z = x^2 + 2xy + yw^2 + \lambda\left(24 - 2x - y - w^2\right) + \upsilon\,(8 - x - w)$. Thus

$$Z_\lambda = 24 - 2x - y - w^2 = 0 \qquad Z_\upsilon = 8 - x - w = 0$$

$$Z_x = 2x + 2y - 2\lambda - \upsilon = 0 \qquad Z_y = 2x + w^2 - \lambda = 0$$

$$Z_w = 2yw - 2\lambda w - \upsilon = 0$$

4. $Z = f(x,y) + \lambda\,[0 - G(x,y)] = f(x,y) - \lambda G(x,y)$. The first-order condition becomes:

$$Z_\lambda = -G(x,y) = 0 \qquad Z_x = f_x - \lambda G_x = 0 \qquad Z_y = f_y - \lambda G_y = 0$$

5. Since the constraint $g = c$ is to prevail at all times in this constrained optimization problem, the equation takes on the sense of an identity, and it follows that dg must be zero. Then it follows that d^2g must be zero, too. In contrast, the equation $dz = 0$ is in the nature of a first-order condition – dz is not identically zero, but is being set equal to zero to locate the critical values of the choice variables. Thus d^2z does not have to be zero as a matter of course.

6. No, the sign of λ^* will be changed. The new λ^* is the negative of the old λ^*.

Exercise 12.3

1.

(a) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 4$, $z^* = \frac{1}{2}$ is a maximum.

(b) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$, $z^* = 36$ is a maximum.

(c) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = -2$, $z^* = -19$ is a minimum.

(d) Since $|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -2$, $z^* = 6\frac{3}{4}$ is a minimum.

2. $|H_1| = \begin{vmatrix} 0 & g_1 \\ g_1 & Z_{11} \end{vmatrix} = -g_1^2 < 0$

3. The zero can be made the last (instead of the first) element in the principal diagonal, with g_1, g_2 and g_3 (in that order appearing in the last column and in the last row.

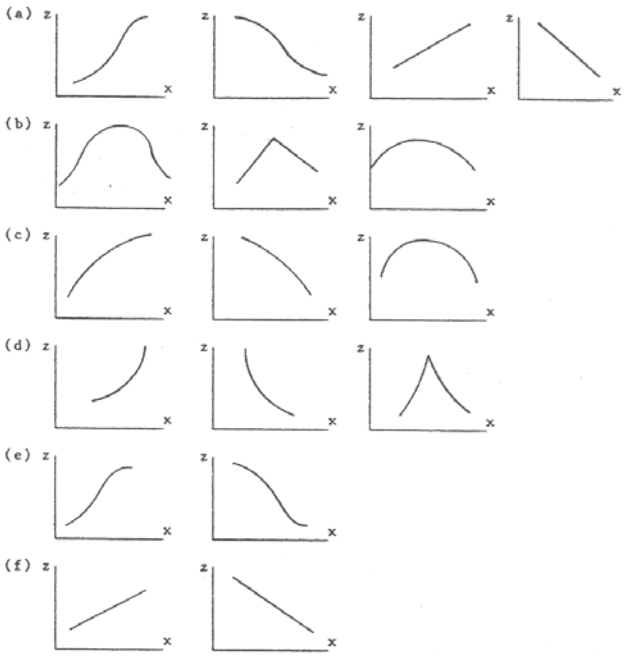
4. $$|\bar{H}| = \begin{vmatrix} 0 & 0 & g_1^1 & g_2^1 & g_3^1 & g_4^1 \\ 0 & 0 & g_1^2 & g_2^2 & g_3^2 & g_4^2 \\ g_1^1 & g_1^2 & Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ g_2^1 & g_2^2 & Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ g_3^1 & g_3^2 & Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ g_4^1 & g_4^2 & Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}$$

   A sufficient condition for maximum z is $|\bar{H}_3| < 0$ and $|\bar{H}_4| = |\bar{H}| > 0$.

   A sufficient condition for minimum z is $|\bar{H}_3| > 0$ and $|\bar{H}| > 0$.

Exercise 12.4

1. Examples of acceptable curves are:



2.

(a) Quasiconcave, but not strictly so. This is because $f(v) = f(u) = a$, and thus $f[\theta u + (1-\theta)v] = a$, which is equal to (not greater than) $f(u)$.

(b) Quasiconcave, and strictly so. In the present case, $f(v) \geq f(u)$ means that $a + bv \geq a + bu$, or $v \geq u$. Moreover, to have u and v distinct, we must actually have $v > u$. Since

$$\begin{aligned} f[\theta u + (1-\theta)v] &= a + b[\theta u + (1-\theta)v] \\ &= a + b[\theta u + (1-\theta)v] + (bu - bu) \\ &= a + bu + b(1-\theta)(v-u) \\ &= f(u)\, b(1-\theta)(v-u) = f(u) + \text{some positive term} \end{aligned}$$

it follows that $f[\theta u + (1-\theta)v] > f(u)$. Hence $f(x) = a + bx$, $(b > 0)$, strictly quasiconcave

.

(c) Quasiconcave, and strictly so. Here, $f(v) \geq f(u)$ means $a + cv^2 \geq a + cu^2$, or $v^2 \leq u^2$ (since $c < 0$). For nonnegative distinct values of u and v, this in turn means $v < u$. Now we have

$$\begin{aligned} f[\theta u + (1-\theta)v] &= a + c[\theta u + (1-\theta)v]^2 + \left(cu^2 - cu^2\right) \\ &= a + cu^2 + c\left\{[\theta u + (1-\theta)v]^2 - u^2\right\} \end{aligned}$$

Using the identity $y^2 - x^2 \equiv (y+x)(y-x)$, we can rewrite the above expression as

$a + cu^2 + c[\theta u + (1-\theta)v + u][\theta u + (1-\theta)v - u]$

$= f(u) + c[(1+\theta)u + (1-\theta)v][(1-\theta)(v-u)]$

$= f(u) +$ some positive term $> f(u)$

Hence $f(x) = a + cx^2$, $(c < 0)$, is strictly quasiconcave.

3. Both $f(x)$ and $g(x)$ are monotonic, and thus quasiconcave. However, $f(x) + g(x)$ displays both a hill and a valley. If we pick $k = 5\frac{1}{2}$, for instance, neither $S^{\geq}$ nor $S^{\leq}$ will be a convex set. Therefore $f(x) + g(x)$ is not quasiconcave.

   (a) This cubic function has a graph similar to Fig. 2.8c, with a hill in the second quadrant and valley in the fourth. If we pick $k = 0$, neither $S^{\geq}$ nor $S^{\leq}$ is a convex set. The function is neither quasiconcave nor quasiconvex.

   (b) This function is linear, and hence both quasiconcave and quasiconvex.

   (c) Setting $x_2 - \ln x_1 = k$, and solving for x_2, we get the isovalue equation $x_2 = \ln x_1 + k$. In the x_1x_2 plane, this plots for each value of k as a log curve shifted upward vertically by the amount of k. The set $S^{\leq} = \{(x_1, x_2) \mid f(x_1, x_2) \leq k\}$ – the set of points on or below the isovalue curve – is a convex set. Thus the function is quasiconvex. (but not quasiconcave).

4.

   (a) A cubic curve contains two bends, and would thus violate both parts of (12.21).

   (b) From the discussion of the cubic total-cost function in Sec. 9.4, we know that if a, c, $d > 0$, $b < 0$, and $b^2 < 3ac$, then the cubic function will be upward-sloping for nonnegative x. Then, by (12.21), it is both quasiconcave and quasiconvex.

5. Let u and v be two values of x, and let $f(v) = v^2 \geq f(u) = u^2$, which implies $v \geq u$. Since $f'(x) = 2x$, we find that

$$f'(u)(v-u) = 2u(v-u) \geq 0$$

$$f'(v)(v-u) = 2v(v-u) \geq 0$$

Thus, by (12.22), the function is both quasiconcave and quasiconvex., confirming the conclusion in Example 1.

6. The set $S^{\leq}$, involving the inequality $xy \leq k$, consists of the points lying on or below a rectangular hyperbola – not a convex set. Hence the function is quasiconvex by (12.21). Alternatively, since $f_x = y$, $f_y = x$, $f_{xx} = 0$, $f_{xy} = 1$, and $f_{yy} = 0$, we have $|B_1| = -y^2 \leq 0$ and $|B_2| = 2xy \geq 0$, which violates the necessary condition (12.25') for quasiconvexity.

7.

   (a) Since $f_x = -2x$, $f_y = -2y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -2$, we have

   $$|B_1| = -4x^2 < 0 \qquad |B_2| = 8\left(x^2+y^2\right) > 0$$

   By (12.26), the function is quasiconcave.

   (b) Since $f_x = -2(x+1)$, $f_y = -2(y+2)$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -2$, we have

   $$|B_1| = -4(x+1)^2 < 0 \qquad |B_2| = 8(x+1)^2 + 8(y+2)^2 > 0$$

   By (12.26), the function is quasiconcave.

Exercise 12.5

1.

   (a) $Z = (x+2)(y+1) + \lambda(130 - 4x - 6y)$

   (b) The first-order condition requires that

   $$Z_\lambda = 130 - 4x - 6y = 0, \qquad Z_x = y + 1 - 4\lambda = 0, \qquad Z_y = x + 2 - 6\lambda = 0$$

   Thus we have $\lambda^* = 3$, $x^* = 16$, and $y^* = 11$.

   (c) $$\left|\bar{H}\right| = \begin{vmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = 48 > 0.$$ Hence utility is maximized.

   (d) No.

2.

(a) $Z = (x+2)(y+1) + \lambda (B - xP_x - yP_y)$

(b) As the necessary condition for extremum, we have

$$\begin{aligned} Z_\lambda &= B - xP_x - yP_y = 0 \qquad \text{or} \qquad & -P_x x - P_y y &= -B \\ Z_x &= y + 1 - \lambda P_x = 0 & -P_x\lambda + y &= -1 \\ Z_y &= x + 2 - \lambda P_y = 0 & -P_y\lambda + x &= -2 \end{aligned}$$

By Cramer's Rule, we can find that

$\lambda^* = \frac{B+2P_x+P_y}{2P_xP_y}$ $\quad x^* = \frac{B-2P_x+P_y}{2P_x}$ $\quad y^* = \frac{B+2P_x-P_y}{2P_y}$

(c) $\left|\bar{H}\right| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & 0 & 1 \\ P_y & 1 & 0 \end{vmatrix} = 2P_xP_y > 0.$ Utility is maximized.

(d) When $P_x = 4$, $P_y = 6$, and $B = 130$, we get $\lambda^* = 3$, $x^* = 16$ and $y^* = 11$. These check with the preceding problem.

3. Yes. $\left(\frac{\partial x^*}{\partial B}\right) = \frac{1}{2P_x} > 0$, $\left(\frac{\partial x^*}{\partial P_x}\right) = -\frac{B+P_y}{2P_x^2} < 0$, $\left(\frac{\partial x^*}{\partial P_y}\right) = \frac{1}{2P_x} > 0$, $\left(\frac{\partial y^*}{\partial B}\right) = \frac{1}{2P_y} > 0$, $\left(\frac{\partial y^*}{\partial P_x}\right) = \frac{1}{P_y} > 0$, $\left(\frac{\partial y^*}{\partial P_y}\right) = -\frac{B+2P_x}{2P_y^2} < 0.$

An increase in income B raises the level of optimal purchases of x and y both; an increase in the price of one commodity reduces the optimal purchase of that commodity itself, but raises the optimal purchase of the other commodity.

4. We have $U_{xx^*} = U_{yy} = 0$, $U_{xy} = U_{yx^*} = 1$, $|J| = \left|\bar{H}\right| = 2P_xP_y$.

$x^* = \frac{(B-2P_x+P_y)}{2P_x}$, and $\lambda^* = \frac{(B+2P_x+P_y)}{2P_xP_y}$. Thus:

(a) $\left(\frac{\partial x^*}{\partial B}\right) = \frac{1}{2P_x}$, and $\left(\frac{\partial y^*}{\partial B}\right) = \frac{1}{2P_y}$.

(b) $\left(\frac{\partial x^*}{\partial P_x}\right) = -\frac{(B+P_y)}{2P_x^2}$, and $\left(\frac{\partial y^*}{\partial P_x}\right) = \frac{1}{P_y}$.

These answers check with the preceding problem.

5. A negative sign for that derivative can mean either that the income effect (T_1) and the substitution effect (T_2) in (12.33') are both negative (normal good), or that the income effect is positive (inferior good) but is overshadowed by the negative substitution effect. The statement is not valid.

6. The optimal utility level can be expressed as $U^* = U^*(x^*, y^*)$. Thus $dU^* = U_x\,dx^* + U_y\,dy^*$, where U_x and U_y are evaluated at the optimum. When U^* is constant, we have $dU^* = 0$, or $U_x\,dx^* + U_y\,dy^* = 0$. From (12.42'), we have $\frac{U_x}{U_y} = \frac{P_x}{P_y}$ at the optimum. Thus we can also express $dU^* = 0$ by $P_x\,dx^* + P_y\,dy^* = 0$, or -$P_x\,dx^* - P_y\,dy^* = 0$.

7.

   (a) No; diminishing marginal utility means only that U_{xx} and U_{yy} are negative, but says nothing about U_{xy}. Therefore we cannot be sure that $\left|\bar{H}\right| > 0$ in (12.32) and $\frac{d^2y}{dx^2} > 0$ in (12.33').

   (b) No; if $\frac{dy^2}{dx^2} > 0$, and hence $\left|\bar{H}\right| > 0$, nothing definite be said about the sign of U_{xx} and U_{yy}, because U_{xy} also appears in $\left|\bar{H}\right|$.

Exercise 12.6

1. (a) $\sqrt{(jx)(jy)} = j = \sqrt{xy}$; homogeneous of degree one.

   (b) $\left[(jx)^2 - (jy)^2\right]^{\frac{1}{2}} = j\left(x^2 - y^2\right)^{\frac{1}{2}}$; homogeneous of degree one.

   (c) Not homogeneous.

   (d) $2jx + jy + 3\sqrt{(jx)(jy)} = j\left(2x + y + 3\sqrt{xy}\right)$; homogeneous of degree one.

   (e) $\frac{(jx)(jy)^2}{jw} + 2(jx)(jw) = j^2\left(\frac{xy^2}{w} + 2xw\right)$; homogeneous of degree two.

   (f) $(jx)^4 - 5(jy)(jw)^3 = j^4\left(x^4 - 5yw^3\right)$; homogeneous of degree four.

2. Let $j = \frac{1}{k}$, then $\frac{Q}{K} = f\left(\frac{K}{K}, \frac{L}{K}\right) = f\left(l, \frac{L}{K}\right) = \psi\left(\frac{L}{K}\right)$. Thus $Q = K\psi\left(\frac{L}{K}\right)$.

   (a) When $MPP_K = 0$, we have $L\frac{\partial Q}{\partial L} = Q$, or $\frac{\partial Q}{\partial L} = \frac{Q}{L}$, or $MPP_L = APP_L$.

   (b) When $MPP_L = 0$, we have $K\frac{\partial Q}{\partial K} = Q$, or $\frac{\partial Q}{\partial K} = \frac{Q}{K}$, or $MPP_K = APP_K$.

3. Yes, they are true:

4.

   (a) $APP_L = \phi(k)$; hence APP_L indeed can be plotted against k.

   (b) $MPP_K = \phi'(k) =$ slope of APP_L.

   (c) $APP_K = \frac{\phi(k)}{k} = \frac{APP_L}{k} = \frac{\text{ordinate of a point on the } APP_L \text{ curve}}{\text{abscissa of that point}}$ =slope of radius vector to the APP_L curve