MATH1104E, Linear Algebra for Engineering or Science,
FALL 2014, TEST # 1

Solutions

Name(print) | Signature | Student Number

Total Pages: 4

Total Marks: 40

INSTRUCTION:
Write your solution in the space provided below the question. If necessary, continue onto the back of the sheet, but remind your marker to look there. Show all your work. No calculator is allowed

1. [**10 marks**] Solve the following system of linear equations if possible. Reduced the augumented matrix to the reduced row echelon form (RREF). If yes, write your solutions in parametric form if there are at least 1 solutions; otherwise, explain why a solution does not exist.

$$\begin{array}{rrrrcr} x_1 & & +2x_3 & +3x_4 & = & 7 \\ 2x_1 & +x_2 & +5x_3 & +6x_4 & = & 19 \\ -2x_1 & -x_2 & -4x_3 & -5x_4 & = & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 7 \\ 2 & 1 & 5 & 6 & 19 \\ -2 & -1 & -4 & -5 & -12 \end{array}\right] \xrightarrow[R_3' = R_3 + 2R_1]{R_2' = R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & -1 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_3' = R_3 + R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

$$\xrightarrow[R_2' = R_2 - R_3]{R_1' = R_1 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -7 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

$$\begin{array}{l} x_1 + x_4 = -7 \\ x_2 - x_4 = -2 \\ x_3 + x_4 = 7 \end{array}$$

$$\begin{cases} x_1 = -7 - t \\ x_2 = -2 + t \\ x_3 = 7 - t \\ x_4 = t, \quad t \in \mathbb{R} \end{cases}$$

2. **[7 marks]** For what values of k and t, if any, will the following system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions?

$$\begin{array}{rrcr} -4x & +2y & = & -6 \\ 2x & +ky & = & 3+t \end{array}$$

$$\left[\begin{array}{cc|c} -4 & 2 & -6 \\ 2 & k & 3+t \end{array}\right] \xrightarrow{-\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{3}{2} \\ 2 & k & 3+t \end{array}\right] \xrightarrow[R_2' = R_2 - 2R_1]{} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & k+1 & t \end{array}\right]$$

a) $k=-1$, $t \neq 0$, no solutions

b) $k \neq -1$, unique solution.

c) $k=-1$, $t=0$, infinitely many solutions

3. [**12 marks**] Let

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) [**10 marks**] Find A^{-1}, if it exists.

$$\left[\begin{array}{ccc|ccc} 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_2' = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3' = R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_1' = R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(b) [**2 marks**] Find a solution $\mathbf{x}$ of the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

$$X = A^{-1} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

4. [6 Marks]

Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -2 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}$$

(a) Check that $(B+C)^T = B^T + C^T$.

(b) Check that $(AB)^T = B^T A^T$.

a) $$(B+C)^T = \left(\begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 4 \\ 1 & 0 \end{bmatrix} \right)^T = \begin{bmatrix} 2 & 0 \\ 1 & 8 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 8 & 1 \end{bmatrix}$$

$\therefore (B+C)^T = B^T + C^T$

$$B^T + C^T = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} 1 & -2 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ -2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 8 & 1 \end{bmatrix}$$

b) $$(AB)^T = \left(\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix}$$

$\therefore (AB)^T = B^T A^T$

5. [**5 marks**] Truth or False questions (1 mark each).

(a) The equation $AX = b$ is homogeneous if $X = 0$ is a solution. T F

(b) If the augumented matrix $[A \ b]$ has a pivot position in every row, then the equation $AX = b$ is inconsistent. T F

(c) A consistent system has infinitely many solutions if and only if there are free variables. T F

(d) For any $m \times n$ matrix A, the matrix AA^T is always symmetric. T F

(e) The product of two invertible $n \times n$ matrices of A and B is invertible and $(AB)^{-1} = A^{-1}B^{-1}$. T F