1. If the coefficient matrix A in a homogeneous system of 16 equations in 20 unknowns is known to have rank 10, how many parameters are there in the general solution?

A. none

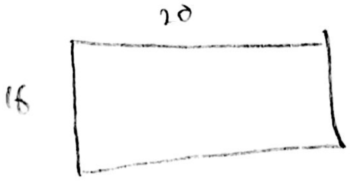
B. 4

C. 6

[✓ D. 10]

E. 16

F. 20



parameters = variables - rank

= 20 - 10

= 10

2. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and B is a $3 \times n$ matrix then the third row of the matrix AB is

A. the same as the second row of A.

B. the same as the first row of B.

C. the same as the second row of B.

D. the sum of the first and the second rows of B.

E. the sum of the first and the third rows of B.

[✓ F. the sum of the second and third rows of B.]

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_1 + c_2 + c_3 \\ c_2 + c_3 \end{bmatrix}$$

3. Find the value(s) of t for which $(1,2,3,t)$ lies in the subspace spanned by $(1,0,1,2)$, $(0,1,1,2)$ and $(1,1,0,2)$.

A. $t = 4$ or 6

[✓ B.] $t = 4$ only

C. $t = 6$ only

D. $t = -2$ or -4

E. $t = 0$ or 2

F. $t = -2$, 0 or 4

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 2 & 2 & 2 & t \end{array}\right] \overset{R_3 - R_1 \to R_3}{\underset{R_4 - 2R_1 \to R_4}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 0 & t-2 \end{array}\right] \overset{R_3 - R_2 \to R_3}{\underset{R_4 - 2R_2 \to R_4}{\sim}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & t-4 \end{array}\right] \overset{R_4 - R_3 \to R_4}{\underset{-\frac{1}{2}R_3 \to R_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t-4 \end{array}\right]$$

Correct answer: C

4. Suppose $e, f \in \mathbb{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcr} 2x & - & 2y & + & ez & = & f \\ x & & & + & z & = & -1 \\ 3x & + & y & + & 2z & = & -1 \end{array}$$

(a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank(A) and rank$[\,A\,|\,b\,]$ for <u>all</u> values of e and f.

$$\left[\begin{array}{ccc|c} 2 & -2 & e & f \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 3 & 1 & 2 & -1 \\ 1 & 0 & 1 & -1 \\ 2 & -2 & e & f \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \\ 2 & -2 & e & f \end{array}\right] \underset{R_3 - 2R_1}{\overset{R_2 - 3R_1}{\sim}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & e-2 & f+2 \end{array}\right] \overset{R_3 + 2R_2 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & e-4 & f\boxed{+4} \end{array}\right] -0.5$$

rank A = 2 if e = 4 ✓

rank A = 3 if e ≠ 4

rank[A|b] = 2 if e = 4 and f = -4 ✓

rank[A|b] = 3 if e ≠ 4 or f ≠ -4

2.5

(Q4 parts (b) and (c) are on the next page...)

4b) Using part (a), find all values of e and f so that this system has

i. a unique solution

Unique soln if # columns = rank(A), so if $e \neq 4$, the system has a unique solution. ✓

ii. infinitely many solutions, or

Infinitely many solutions if rank([A|b]) < # of columns(A), so if $e = 4$ and $f = -4$, the system will have infinitely many solutions. ✓

1.5

iii. no solutions

If the system is inconsistent then it will have no solutions. So if $e = 4$ and $f \neq -4$, the system will be inconsistent, as the bottom row would make a degenerate equation. ✓

4c) In case b(ii) above, give a complete geometric description of the set of solutions.

$$\begin{array}{c} \begin{matrix} & & s & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = -1 - s$

$x_2 = 2 + s$, $s \in \mathbb{R}$

$0 = 0$

$x_3 = s$

1

∴ It is the line through the pt. (-1,2) with directional vector (-1,1

5

5b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\begin{array}{cccccccc} & & & & s & t & & \\ 1 & 0 & 0 & 0 & -1 & 0 & \mid & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & \mid & 50 \\ 0 & 0 & 1 & 0 & -1 & 1 & \mid & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & \mid & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mid & 0 \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$x_1 = -40 + s$

$x_2 = 50 + s - t$ $\quad s, t \in \mathbb{R}$

$x_3 = 30 + s - t$

$x_4 = 60 + s - t$ $\quad$ ✓ 2

$x_5 = s$

$x_6 = t$

5c) If ED were closed due to roadwork, find the minimum flow along AC, **using your results from (b).**

(You must justify all your answers.)

$x_3 = 0$, find x_6

$x_1 \geq 0, -40 + s \geq 0 \longrightarrow s \geq 40$ $\quad$ So, min value of s is $\textcircled{40}$ ✓

$x_2 \geq 0, 50 + s - t \geq 0 \rightarrow s \geq -50 + t$ $\quad$ $s + 50 \geq t$

$x_3 \geq 0, 30 + s - t \geq 0 \longrightarrow s \geq -30 + t$ $\quad$ $s + 30 \geq t$ $\quad$ 1

$x_4 \geq 0, 60 + s - t \geq 0 \longrightarrow s \geq -60 + t$ $\quad$ $s + 60 \geq t$

$x_5 \geq 0, s \geq 0$ $\quad$ $t \leq 90$

$x_6 \geq 0, t \geq 0$ $\quad$ $t \leq 70$ $\quad$ t must be ≤ 70 and also ≥ 0

$t \leq 100$

$\boxed{x_6 = 70}$ $\quad$ $\boxed{x_3 = 0}$ $\quad$ $x_3 = 30 + s - t$

$x_3 = 30 + 40 - t$

∴ The minimum flow along AC is ✓70 $\quad$ $x_3 = 70 - t$

$x_3 = 70 - 70$ $\quad$ $\boxed{\text{So } t = 70}$ $\quad$ 6

$x_3 = 0$ $\quad$ and this satisfies t's constraints

6. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers, matrices, or functions, as is appropriate!</u>
- If you say the statement is always true, you must give a clear explanation.

(a) The kernel of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has dimension 1.

A

$$\text{Ker}(A) = \begin{array}{c} \begin{array}{cccc} & s & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$x_1 = -s$

$x_2 = s$, $s \in \mathbb{R}$ $\longrightarrow$ $\{(-1, 1, 0)\}$ is the span of $\text{ker}(A)$

$x_3 = 0$

and only has one element so it does have a dimension of 1. ✓

1.5

ANSWER TRUE ✓

(b) If a linear system is inconsistent, it cannot be homogeneous.

In order for a linear system to be homogeneous, it must be consistent and $Ax = 0$. ✓

1.5

So if $A = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 0$, $1 = 0$, is inconsistent.

remember the last part, while true, does not prove anything!

ANSWER TRUE ✓

6 (cont.)

(c) If A is a 2×2 matrix and $A^2 = 0$, then $A = 0$.

Let $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ then, $A^2 = \begin{array}{cc} & \begin{bmatrix}0&1\\0&0\end{bmatrix} \\ \begin{bmatrix}0&1\\0&0\end{bmatrix} & \begin{bmatrix}0&0\\0&0\end{bmatrix}\end{array}$ $A^2 = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

Since $A \neq 0$, (but $A^2 = 0$) this statement is false ✓

1.5

ANSWER FALSE ✓

(d) If the $[\,A\,|\,b\,]$ is a linear system and A is a 2×3 matrix, and A has a row of zeros, then $[\,A\,|\,b\,]$ has infinitely many solutions.

Let $A = \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$, $[A|b] = \left[\begin{array}{ccc|c}1&0&0&1\\0&0&0&1\end{array}\right]$

$x_1 = 1$

$0 = 1$ *Degenerate equation*

and $b = \begin{bmatrix}1\\1\end{bmatrix}$

1.5

This particular augmented matrix has no solutions, so this statement is false. ✓

ANSWER FALSE ✓

6