MAT 1341F Test 3

Fall 2015

November 12

Professor: Charles Starling

Family Name: Key

First Name:

Student Number:

Enter Multiple Choice Answers Here	
1	D
2	F
3	C

DGD TA: ☐ Evelyn
☐ Cameron

Marker's Use Only	
4	
5	
6	
7 [Bonus]	
Total	

PLEASE READ THESE INSTRUCTIONS VERY CAREFULLY.

1. You have 80 minutes to complete this exam.
2. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All cybernetic implants not necessary for life-support must be disabled at the beginning of the exam.
3. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 4 to 7, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
4. Questions 1 to 3 are multiple choice. These questions are worth 1 point each and no part marks will be given. **Please record your answers in the table above.**
5. Questions 4 to 6 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
6. Question 7 is a challenging bonus question and is worth 3 points. It is *much* more difficult to obtain marks in the bonus question, so spend your time accordingly. You can earn 100% without attempting Q.7.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. If the coefficient matrix A in a homogeneous system of 16 equations in 20 unknowns is known to have rank 10, how many parameters are there in the general solution?

A. none

B. 4

C. 6

D. 10

E. 16

F. 20

param = # cols A − rank(A)
$= 20 - 10$
$= 10$

2. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and B is a $3 \times n$ matrix then the third row of the matrix AB is

A. the same as the second row of A.

B. the same as the first row of B.

C. the same as the second row of B.

D. the sum of the first and the second rows of B.

E. the sum of the first and the third rows of B.

F. the sum of the second and third rows of B.

$B = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$AB: \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_1+r_2+r_3 \\ r_2+r_3 \end{bmatrix}$

3. Find the value(s) of t for which $(1, 2, 3, t)$ lies in the subspace spanned by $(1, 0, 1, 2)$, $(0, 1, 1, 2)$ and $(1, 1, 0, 2)$.

A. $t = 4$ or 6

B. $t = 4$ only

(C.) $t = 6$ only

D. $t = -2$ or -4

E. $t = 0$ or 2

F. $t = -2, 0$ or 4

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 2 & 2 & 2 & t \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 0 & t-2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & t-6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t-6 \end{array}\right]$$

only consistent if $t = 6$

4. Suppose $e, f \in \mathbb{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcr} 2x & - & 2y & + & ez & = & f \\ x & & & + & z & = & -1 \\ 3x & + & y & + & 2z & = & -1 \end{array}$$

(a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find $\text{rank}(A)$ and $\text{rank}[\,A\,|\,b\,]$ for all values of e and f.

$$\left[\begin{array}{ccc|c} 2 & -2 & e & f \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \\ 2 & -2 & e & f \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & e-2 & f+2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & e-4 & f+6 \end{array}\right]$$

$$\text{rank}(A) = \begin{cases} 2 & \text{if } e=4 \\ 3 & \text{if } e \neq 4 \end{cases}$$

$$\text{rank}([A|b]) = \begin{cases} 2 & \text{if } e=4 \text{ and } f=-6 \\ 3 & \text{if } e\neq 4 \text{ or } f \neq -6. \end{cases}$$

(Q4 parts (b) and (c) are on the next page...)

4b) Using part (a), find all values of e and f so that this system has

i. a unique solution

unique solution $\iff$ rank (A) = rank $[A|b]$ = # cols $A = 3$

$\iff e \neq 4$.

ii. infinitely many solutions, or

∞ solutions $\iff$ rank(A) = rank $[A|b]$ < # cols $A = 3$

$\iff$ rank (A) = rank $[A|b] = 2$

$\iff e = 4$ and $f = -6$

iii. no solutions

no solution $\iff$ rank$(A) \neq$ rank $[A|b]$

$\iff$ rank $(A) = 2$, rank $(A|b) = 3$

$\iff e = 4$ and $f \neq -6$.

4c) In case b(ii) above, give a complete geometric description of the set of solutions.

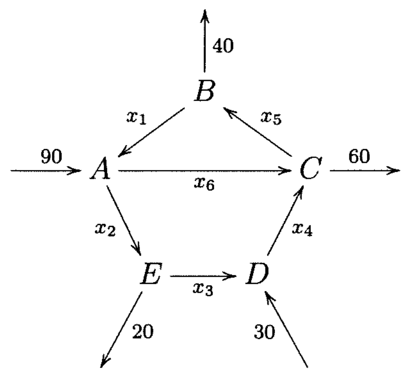
$e = 4$, $f = -6$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x &= -1-t \\ y &= 2+t \\ z &= t \end{aligned}$$

set of solutions $= \left\{ \begin{bmatrix} -1-t \\ 2+t \\ t \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

This is a line through $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ with direction $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

5. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) Write down a system of linear equations which describes the the traffic flow, together with all the constraints on the variables x_i, $i = 1, \ldots, 6$.

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

IN = OUT

A: $x_1 + 90 = x_2 + x_6$

B: $x_5 = x_1 + 40$

C: $x_4 + x_6 = x_5 + 60$

D: $x_3 + 30 = x_4$

E: $x_2 = x_3 + 20$

$x_i \geq 0$ for all i

x_i integer for all i

5b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 50 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$x_1 = s - 40$
$x_2 = s - t + 50$
$x_3 = s - t + 30$
$x_4 = s - t + 60$
$x_5 = s$
$x_6 = t$

5c) If ED were closed due to roadwork, find the minimum flow along AC, **using your results from (b).**

(You must justify all your answers.)

ED closed $\Rightarrow$ $x_3 = 0$ $\Rightarrow$ $s = t - 30$

$\Rightarrow$ $x_1 = t - 70 \geq 0 \rightarrow t \geq 70$
$x_2 = 20 \geq 0$ ✓
$x_3 = 0 \geq 0$ ✓
$x_4 = 30 \geq 0$ ✓
$x_5 = t - 30 \geq 0 \Rightarrow t \geq 30$
$x_6 = t \geq 0 \Rightarrow t \geq 0$

All must be positive simultaneously, so we need

$\boxed{t \geq 70}$

6. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers, matrices, or functions, as is appropriate!</u>
- If you say the statement is always true, you must give a clear explanation.

(a) The kernel of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has dimension 1.

$\dim \ker(A) = \#\text{cols } A - \operatorname{rank}(A)$
$= 3 - 2 = 1$

ANSWER True

(b) If a linear system is inconsistent, it cannot be homogeneous.

All homogeneous systems are consistent.

ANSWER True

6 (cont.)

(c) If A is a 2×2 matrix and $A^2 = 0$, then $A = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$$

$$A^2 = 0$$

ANSWER False

(d) If the $[\,A\,|\,b\,]$ is a linear system and A is a 2×3 matrix, and A has a row of zeros, then $[\,A\,|\,b\,]$ has infinitely many solutions.

$$[A \,|\, b] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ has no solution.

ANSWER False

7. [Challenge/Bonus]

Suppose A is a nonzero 3×3 matrix with $A^t = -A$, where A^t denotes the transpose of A.

Prove that $\operatorname{rank}(A) = 2$.

(Your proof must work for all nonzero 3×3 matrices A with $A^t = -A$: do not choose a particular matrix.)

If $A^t = -A$, then $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$.

Since $A \neq 0$, we have 3 cases

1) $a \neq 0$: $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \sim \begin{bmatrix} -a & 0 & c \\ -b & -c & 0 \\ 0 & a & b \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -c/a \\ 0 & -c & -\frac{bc}{a} \\ 0 & a & b \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -c/a \\ 0 & 1 & b/a \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -c/a \\ 0 & 1 & b/a \\ 0 & 0 & 0 \end{bmatrix}$ rank(A)=2

2) $b \neq 0$ & $a = 0$: $A \sim \begin{bmatrix} -b & -c & 0 \\ 0 & 0 & b \\ 0 & 0 & c \end{bmatrix} \sim \begin{bmatrix} 1 & c/b & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c \end{bmatrix} \sim \begin{bmatrix} 1 & c/b & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rank(A)=2

3) $a=0, b=0, c \neq 0$ $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rank(A) = 2.