3.21: Sln: $\because$ $x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$.

and $a_1=a_{-1}^*=j$, $a_5=a_{-5}=2$, $a_k=0$ for other k.

$\therefore x(t)=a_1 e^{j(2\pi/T)t}+a_{-1}e^{-j(2\pi/T)t}+a_5 e^{j5(2\pi/T)t}+a_{-5}e^{-j5(2\pi/T)t}$

$=je^{j(2\pi/8)t}+(-j)e^{-j(2\pi/8)t}+2e^{j5(2\pi/8)t}+2e^{-j5(2\pi/8)t}$

since Euler's formula: $e^{jx}=\cos x+j\sin x$.

and $\frac{e^{jx}+e^{-jx}}{2}=\cos x$, $\frac{e^{jx}-e^{-jx}}{2j}=\sin x$,

$x(t)=j\left(e^{j(2\pi/8)t}-e^{-j(2\pi/8)t}\right)+2\left(e^{j5(2\pi/8)t}+e^{-j5(2\pi/8)t}\right)$

$=j\cdot 2j\sin\left(\frac{2\pi}{8}t\right)+2\cdot 2\cos\left(5\cdot\frac{2\pi}{8}t\right)$

$=-2\sin\left(\frac{\pi}{4}t\right)+4\cos\left(\frac{5\pi}{4}t\right)$

$=-2\cos\left(\frac{\pi}{4}t-\frac{\pi}{2}\right)+4\cos\left(\frac{5\pi}{4}t\right)$.

Now $T=8 \Rightarrow \omega_0=\frac{2\pi}{8}=\frac{\pi}{4}$.

So $x(t)$ in the form $\sum_{k=0}^{\infty} A_k\cos(W_k t+\varphi_k)$,

where $\begin{cases} A_1=-2,\ W_1=1\cdot\omega_0=\frac{\pi}{4},\ \varphi_1=-\frac{\pi}{2}. \\ A_5=4,\ W_5=5\cdot W_0=\frac{5\pi}{4},\ \varphi_5=0. \end{cases}$

3.22(d) First, since $x(t+2)=x(t)$, $T=2$. $\Rightarrow \omega_0=\frac{2\pi}{T}=\pi$.

Then $a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$.

We pick one period over which the integral is carried on, for example, $t\in(-0.5,1.5)$. You should avoid to choose $0<t<2$, because it creates uncertainty: $\delta(t)$ and $\delta(t-2)$ at the boundary of the interval.

$$\Rightarrow a_k=\frac{1}{T}\int_{-0.5}^{1.5}x(t)e^{-jk\omega_0 t}dt$$

in $-0.5<t<1.5$, $x(t)=\delta(t)-2\delta(t-1)$.

$$=\frac{1}{2}\int_{-0.5}^{1.5}(\delta(t)-2\delta(t-1))e^{-jk\pi t}dt$$

$$=\frac{1}{2}\int_{-0.5}^{1.5}\delta(t)e^{-jk\pi t}dt-\frac{1}{2}\cdot 2\int_{-0.5}^{1.5}\delta(t-1)e^{-jk\pi t}dt$$

$$=\frac{1}{2}e^{-jk\pi\cdot 0}\underbrace{\int_{-0.5}^{1.5}\delta(t)dt}_{=1}-\frac{1}{2}\cdot 2e^{-jk\pi\cdot 1}\underbrace{\int_{-0.5}^{1.5}\delta(t-1)dt}_{=1}$$

$$=\frac{1}{2}-\underbrace{(e^{-j\pi})}_{=\cos(-\pi)+j\sin(-\pi)=-1}{}^{k}$$

$$=\frac{1}{2}-(-1)^k$$

$$x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{+\infty}\left(\frac{1}{2}-(-1)^k\right)e^{jk\pi t}$$

(f) Since $x(t+3)=x(t)$, $T=3 \Rightarrow \omega_0=\frac{2\pi}{T}=\frac{2\pi}{3}$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

For example, we pick t from 0 to 3.

$$a_k=\frac{1}{3}\int_0^3 x(t)e^{-jk\frac{2\pi}{3}t}dt$$

Since $x(t)=2$ for $0<t<1$; $x(t)=1$ for $1<t<2$; $x(t)=0$ for $2<t<3$.

$$a_k=\frac{1}{3}\int_0^1 2e^{-jk\frac{2\pi}{3}t}dt+\frac{1}{3}\int_1^2 1\cdot e^{-jk\frac{2\pi}{3}t}dt$$

$$=\begin{cases}\frac{2}{3}\cdot t\Big|_0^1+\frac{1}{3}t\Big|_1^2 & (k=0)\\ \frac{2}{3}\cdot\frac{e^{-jk\frac{2\pi}{3}t}}{(-jk\frac{2\pi}{3})}\Big|_0^1+\frac{1}{3}\cdot\frac{e^{-jk\frac{2\pi}{3}t}}{(-j\frac{2\pi}{3}k)}\Big|_1^2 & (k\neq 0)\end{cases}$$

$$=\begin{cases}1 & (k=0)\\ \frac{2-e^{-jk\frac{2\pi}{3}}-e^{-jk\frac{4\pi}{3}}}{jk2\pi}, & (k\neq 0)\quad \star\end{cases}$$

$\Rightarrow x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}$ where a_k is above.

We note that $\star$ $(k\neq 0)$, a_k can be reduced to

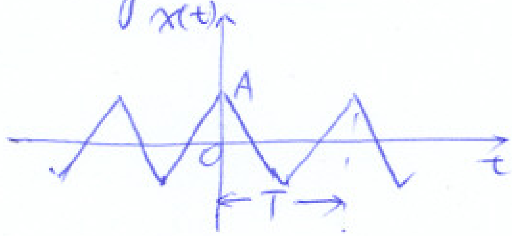
$$\frac{(1-e^{-jk\frac{2\pi}{3}})+(1-e^{-jk\frac{4\pi}{3}})}{2jk\pi}$$

$$\frac{e^{-jk\frac{\pi}{3}}(e^{jk\frac{\pi}{3}}-e^{-jk\frac{\pi}{3}})+e^{-jk\frac{2\pi}{3}}(e^{jk\frac{2\pi}{3}}-e^{-jk\frac{2\pi}{3}})}{2jk\pi}$$

$$=\frac{e^{-jk\frac{\pi}{3}}\cdot 2j\sin(k\frac{\pi}{3})+e^{-jk\frac{2\pi}{3}}\cdot 2j\sin(\frac{2\pi}{3})}{2jk\pi}$$

$$=\frac{e^{-jk\frac{\pi}{3}}\sin(k\frac{\pi}{3})+e^{-jk\frac{2\pi}{3}}\cdot\sin(\frac{2\pi}{3})}{k\pi}\qquad (k\neq 0)$$

Triangular Wave $\omega_0 = \frac{2\pi}{T}$, $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$



Pick t from $-\frac{T}{2}$ to $\frac{T}{2}$ to do integral.

$$x(t) = \begin{cases} \frac{4A}{T}t + A, & -\frac{T}{2} < t < 0 \\ -\frac{4A}{T}t + A, & 0 < t < \frac{T}{2} \end{cases}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{0}\left(\frac{4A}{T}t + A\right)e^{-jk\omega_0 t}dt + \frac{1}{T}\int_0^{T/2}\left(-\frac{4A}{T}t + A\right)e^{-jk\omega_0 t}dt$$

$$= \frac{4A}{T^2}\int_{-T/2}^{0} t e^{-jk\omega_0 t}dt + \frac{A}{T}\int_{-T/2}^{0} e^{-jk\omega_0 t}dt + \left(-\frac{4A}{T^2}\right)\int_0^{T/2} t \cdot e^{-jk\omega_0 t}dt + \frac{A}{T}\int_0^{T/2} e^{-jk\omega_0 t}dt$$

If $k=0$, $a_k = \frac{4A}{T^2}\int_{-T/2}^{0} t\,dt + \frac{A}{T}\int_{-T/2}^{0} dt - \frac{4A}{T^2}\int_0^{T/2} t\,dt + \frac{A}{T}\int_0^{T/2} dt$

$$= \frac{4A}{T^2}\cdot\frac{t^2}{2}\Big|_{-T/2}^{0} + \frac{A}{T}t\Big|_{-T/2}^{0} - \frac{4A}{T^2}\cdot\frac{t^2}{2}\Big|_0^{T/2} + \frac{A}{T}t\Big|_0^{T/2} = 0.$$

If $k \neq 0$, $a_k = \frac{4A}{T^2}\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)^2}(-jk\omega_0 t - 1)\Big|_{-T/2}^{0} + \frac{A}{T}\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)}\Big|_{-T/2}^{0}$

$$+ \left(-\frac{4A}{T^2}\right)\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)^2}(-jk\omega_0 t - 1)\Big|_0^{T/2} + \frac{A}{T}\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)}\Big|_0^{T/2}$$

$$= \frac{4A}{T^2}\frac{1}{(-jk\omega_0)^2}(-1) - \frac{4A}{T^2}\frac{e^{+jk\omega_0\frac{T}{2}}}{(-jk\omega_0)^2}\left(+jk\omega_0\frac{T}{2} - 1\right) + \cancel{\frac{A}{T}\frac{1}{(-jk\omega_0)}} - \frac{A}{T}\frac{e^{jk\omega_0\frac{T}{2}}}{(-jk\omega_0)}$$

$$+ \left(-\frac{4A}{T^2}\right)\frac{e^{-jk\omega_0\frac{T}{2}}}{(-jk\omega_0)^2}\left(-jk\omega_0\frac{T}{2} - 1\right) - \left(-\frac{4A}{T^2}\right)\frac{1}{(-jk\omega_0)^2}(-1) + \frac{A}{T}\frac{e^{-jk\omega_0\frac{T}{2}}}{(-jk\omega_0)} - \cancel{\frac{A}{T}\frac{1}{(-jk\omega_0)}}$$

$$= -\frac{4A}{T^2}\frac{1}{(-jk\omega_0)^2} - \frac{4A}{T^2}\cdot jk\omega_0\frac{T}{2}\cdot\frac{e^{jk\omega_0 T/2}}{(-jk\omega_0)^2} + \frac{4A}{T^2}\frac{e^{jk\omega_0 T/2}}{(-jk\omega_0)^2} - \frac{A}{T}\frac{e^{jk\omega_0 T/2}}{(-jk\omega_0)}$$

$$+ \frac{4A}{T^2}jk\omega_0\frac{T}{2}\frac{e^{-jk\omega_0\frac{T}{2}}}{(-jk\omega_0)^2} + \frac{4A}{T^2}\frac{e^{-jk\omega_0\frac{T}{2}}}{(-jk\omega_0)^2} - \frac{4A}{T^2}\frac{1}{(-jk\omega_0)^2} + \frac{A}{T}\frac{e^{-jk\omega_0 T/2}}{(-jk\omega_0)}$$

$$= -\frac{4A}{T^2}\cdot jk\omega_0\frac{T}{2}\frac{1}{(-jk\omega_0)^2}\left(e^{jk\omega_0 T/2} - e^{-jk\omega_0 T/2}\right) + \frac{4A}{T^2}\cdot\frac{1}{(-jk\omega_0)^2}\left(e^{jk\omega_0\frac{T}{2}} + e^{-jk\omega_0\frac{T}{2}}\right)$$

$$- \frac{A}{T}\frac{1}{(-jk\omega_0)}\left(e^{jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\frac{T}{2}}\right) - 2\left(\frac{4A}{T^2}\frac{1}{(-jk\omega_0)^2}\right)$$

$$= -\frac{4A}{T^2}\frac{\frac{T}{2}}{jk\omega_0}\cdot 2j\sin\left(k\omega_0\frac{T}{2}\right) + \frac{4A}{T^2}\frac{1}{(-jk\omega_0)^2}\, 2\cos\left(k\omega_0\frac{T}{2}\right)$$

$$+ \frac{A}{T}\frac{1}{jk\omega_0}2j\sin\left(k\omega_0\frac{T}{2}\right) - \frac{8A}{T^2}\frac{1}{(-jk\omega_0)^2}$$

$$= \cancel{-\frac{2A}{Tk\omega_0}\sin\left(k\omega_0\frac{T}{2}\right)} - \frac{8A}{T^2}\frac{1}{(-jk\omega_0)^2}\left(1 - \cos\left(k\omega_0\frac{T}{2}\right)\right)$$

$\shortparallel$
0. $(\omega_0 T = 2\pi)$

Since $\omega_0 T = 2\pi$

$$k \neq 0, \; a_k = \frac{8A}{T^2 k^2 \omega_0^2}\left(1 - \cos(k\pi)\right)$$

$$= \frac{8A}{(2\pi)^2 k^2}\left(1 - \cos(k\pi)\right)$$

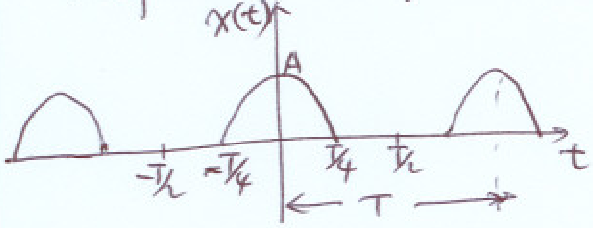
$$= \frac{2A}{\pi^2 k^2}\left(1 - (-1)^k\right)$$

$\Rightarrow$ k even, $a_k = 0$.

k odd, $a_k = \dfrac{4A}{\pi^2 k^2}$.

$$\Rightarrow \quad a_k = \begin{cases} 0, & k \text{ even.} \\ \dfrac{4A}{\pi^2 k^2}, & k \text{ odd.} \end{cases}$$

Half-wave rectified cosine. $\omega_0 = \frac{2\pi}{T}$, $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$



Pick t from $-T/2$ to $T/2$.

$x(t) = A\cos(\omega_0 t)$, $-T/4 < t < T/4$.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-T/4}^{T/4} A\cos(\omega_0 t) e^{-jk\omega_0 t} dt = \frac{A}{T}\int_{-T/4}^{T/4} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-jk\omega_0 t} dt$$

$$= \frac{A}{2T}\int_{-T/4}^{T/4} e^{-j(k-1)\omega_0 t} dt + \frac{A}{2T}\int_{-T/4}^{T/4} e^{-j(k+1)\omega_0 t} dt$$

If $k=1$, $a_k = \frac{A}{2T}\int_{-T/4}^{T/4} dt + \frac{A}{2T}\int_{-T/4}^{T/4} e^{-j2\omega_0 t} dt = \frac{A}{2T} t \Big|_{-T/4}^{T/4} + \frac{A}{2T} \frac{e^{-j2\omega_0 t}}{(-j2\omega_0)}\Big|_{-T/4}^{T/4} = \frac{A}{4} + \frac{A\sin(2\omega_0 \frac{T}{4})}{2\omega_0 T}$

since $\omega_0 T = 2\pi$ $= \frac{A}{4}$.

If $k=-1$, $a_k = \frac{A}{2T}\int_{-T/4}^{T/4} e^{j2\omega_0 t} dt + \frac{A}{2T}\int_{-T/4}^{T/4} dt = \frac{A}{2T} \frac{e^{j2\omega_0 t}}{j2\omega_0}\Big|_{-T/4}^{T/4} + \frac{A}{2T} t\Big|_{-T/4}^{T/4} = \frac{A}{4}$.

$(k \neq \pm 1)$: $a_k = \frac{A}{2T} \cdot \frac{e^{-j(k-1)\omega_0 t}}{(-j(k-1)\omega_0)}\Big|_{-T/4}^{T/4} + \frac{A}{2T} \cdot \frac{e^{j(k+1)\omega_0 t}}{(-j(k+1)\omega_0)}\Big|_{-T/4}^{T/4}$

$$= -\frac{A}{2T} \frac{e^{-j(k-1)\omega_0 \frac{T}{4}} - e^{+j(k-1)\omega_0 \frac{T}{4}}}{j(k-1)\omega_0} - \frac{A}{2T} \frac{e^{-j(k+1)\omega_0 \frac{T}{4}} - e^{j(k+1)\omega_0 \frac{T}{4}}}{j(k+1)\omega_0}$$

$$= \frac{A}{2T} \frac{2j\sin((k-1)\omega_0 \frac{T}{4})}{j(k-1)\omega_0} + \frac{A}{2T} \frac{2j\sin((k+1)\omega_0 \frac{T}{4})}{j(k+1)\omega_0}$$

$$= \frac{A\sin((k-1)\omega_0 \frac{T}{4})}{T(k-1)\omega_0} + \frac{A\sin((k+1)\omega_0 \frac{T}{4})}{T(k+1)\omega_0}$$

Since $\omega_0 T = 2\pi$

$$a_k = \frac{A\sin(\frac{\pi}{2}(k-1))}{2\pi(k-1)} + \frac{A\sin(\frac{\pi}{2}(k+1))}{2\pi(k+1)}$$

$$= \begin{cases} k \text{ odd}, k \neq 1, -1, a_k = 0. \\ k \text{ even}, \frac{A(-1)^{\frac{k}{2}+1}}{2\pi(k-1)} + \frac{A \cdot (-1)^{\frac{k}{2}}}{2\pi(k+1)} \end{cases}$$

$$= \frac{A(-1)^{\frac{k}{2}}(k-1) - A(-1)^{\frac{k}{2}}(k+1)}{2\pi(k-1)(k+1)}$$

$$= -\frac{2A(-1)^{\frac{k}{2}}}{2\pi(k^2-1)}$$

$$= \frac{A}{\pi} \cdot \frac{(-1)^{\frac{k}{2}+1}}{k^2-1}$$

So,

$$a_k = \begin{cases} \frac{A}{4}, & k = \pm 1. \\ 0, & k \text{ odd but } k \neq \pm 1. \\ \frac{A}{\pi} \cdot \frac{(-1)^{\frac{k}{2}+1}}{k^2-1}, & k \text{ even} \end{cases}$$