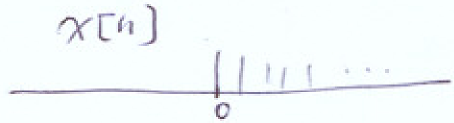
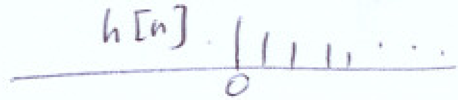
2.21 (a) $x[n] = \begin{cases} \alpha^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$



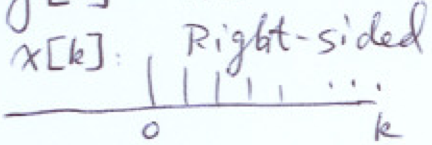
Right-sided

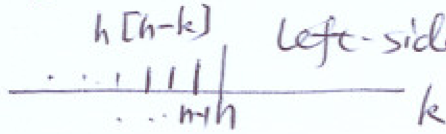
$h[n] = \begin{cases} \beta^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$



Right-sided.

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$





Regions of overlap:

① non-overlap: $n < 0$, $y[n] = 0$.

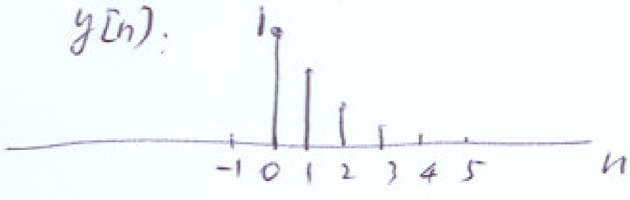
② overlap: $n \ge 0$, $y[n] = \sum_{k=0}^{n} x[k] \cdot h[n-k]$

$$= \sum_{k=0}^{n} \alpha^k \cdot \beta^{n-k}$$

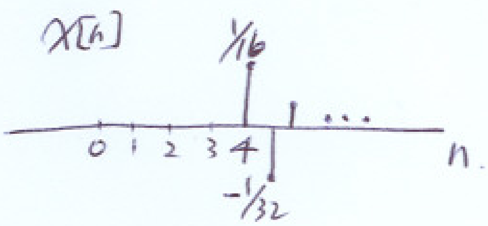
$$= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}.$$

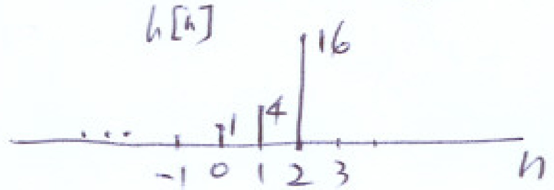
Combine ① and ②, $y[n] = \left(\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right) u[n]$ for $\alpha \neq \beta$.

$\alpha = 1/2$, $\beta = 1/4$

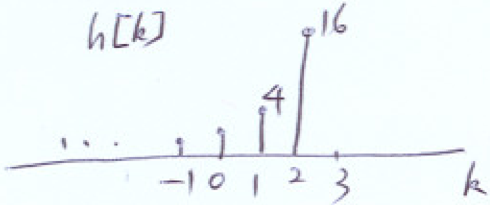


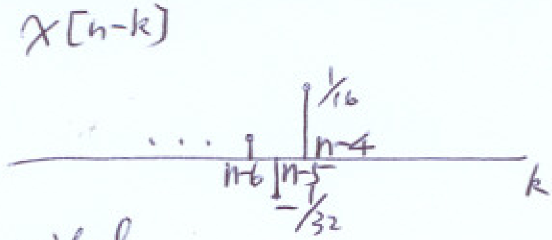
(c) $x[n] = (-\frac{1}{2})^n u[n-4]$ $\qquad$ $h[n] = 4^n u[2-n]$





$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$

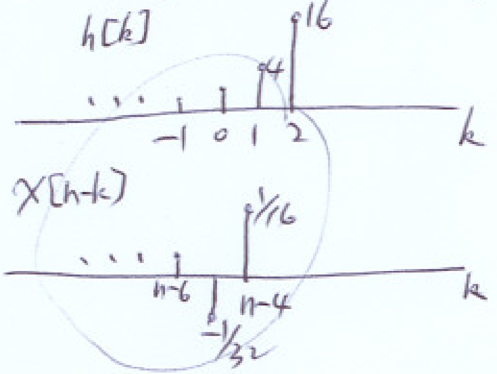




all left-sided

Regions of overlap:

① $n-4 \leq 2 \Leftrightarrow n \leq 6$,



$$y[n] = \sum_{k=-\infty}^{n-4} h[k] \cdot x[n-k]$$

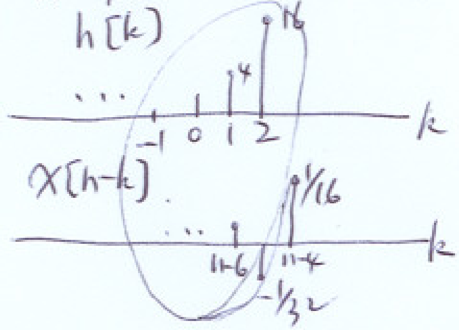
$$= \sum_{k=-\infty}^{n-4} 4^k \underbrace{u[2-k]}_{=1} \cdot \left(-\tfrac{1}{2}\right)^{n-k} \underbrace{u[n-k-4]}_{=1}$$

Remember $k \leq n-4$, and $n \leq 6$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{n-4} 4^k \cdot \left(-\tfrac{1}{2}\right)^{n-k}$$

$$= \left(-\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{n-4} (-8)^k = \left(\tfrac{8}{9}\right)\left(\tfrac{1}{8}\right)^4 4^n.$$

② $n-4 > 2 \Rightarrow n > 6$.

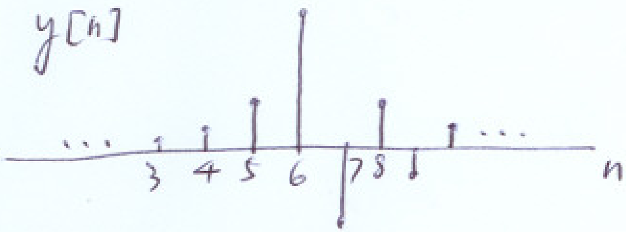


$$y[n] = \sum_{k=-\infty}^{2} h[k] \cdot x[n-k]$$

$$= \sum_{k=-\infty}^{2} 4^k \underbrace{u[2-k]}_{=1} \cdot \left(-\tfrac{1}{2}\right)^{n-k} \underbrace{u[n-k-4]}_{=1}$$

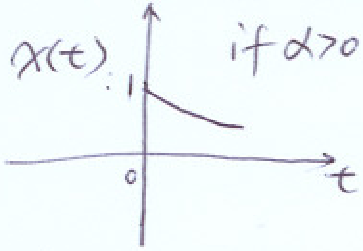
Remember $n>6$ and $k \leq 2$

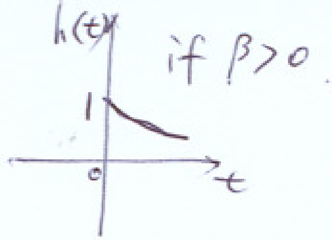
$$\Rightarrow y[n] = \sum_{k=-\infty}^{2} 4^k \left(-\tfrac{1}{2}\right)^{n-k} = \left(-\tfrac{1}{2}\right)^n \sum_{k=-\infty}^{2} (-8)^k = \left(\tfrac{8}{9}\right) 64 \left(-\tfrac{1}{2}\right)^n$$



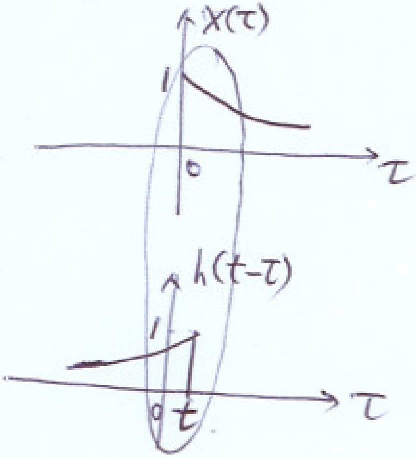
$$\therefore y[n] = \begin{cases} \left(\tfrac{8}{9}\right)\left(\tfrac{1}{8}\right)^4 4^n, & n \leq 6 \\ \left(\tfrac{8}{9}\right) 64 \left(-\tfrac{1}{2}\right)^n, & n > 6 \end{cases}$$

2.22(a)





$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



1. When $\alpha \neq \beta$

Regions of overlap:

① $t<0$: non-overlap.

$y(t)=0$.

② $t>0$: overlap.

$$y(t)=\int_0^t x(\tau)h(t-\tau)=\int_0^t e^{-\alpha\tau}\underbrace{u(\tau)}_{\substack{1\,(\tau>0)}}e^{-\beta(t-\tau)}\underbrace{u(t-\tau)}_{\substack{=1\\(\tau<t)}}d\tau$$

$$=\int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau$$

$$=e^{-\beta t}\int_0^t e^{(\beta-\alpha)\tau}d\tau=e^{-\beta t}\frac{(e^{(\beta-\alpha)t}-1)}{\beta-\alpha}.$$

Combine ① and ②

$$y(t)=e^{-\beta t}\frac{(e^{(\beta-\alpha)t}-1)}{\beta-\alpha}u(t).$$

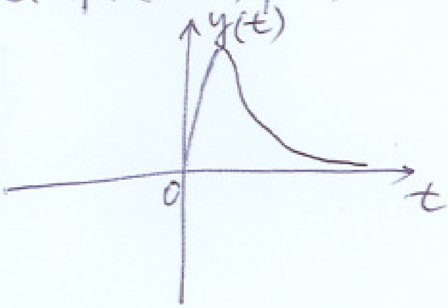
2. When $\alpha=\beta$.

① $t<0$: (non-overlap) $y(t)=0$.

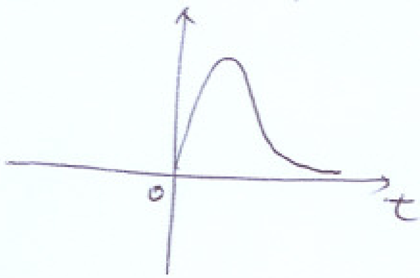
② $t>0$ (overlap) $y(t)=\int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau=te^{-\beta t}$.

Combine ①, ②, $y(t)=te^{-\beta t}u(t)$

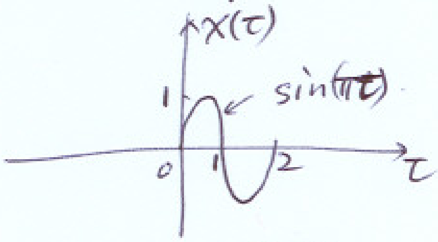
$\alpha\neq\beta$: ($\alpha=2$, $\beta=1$)

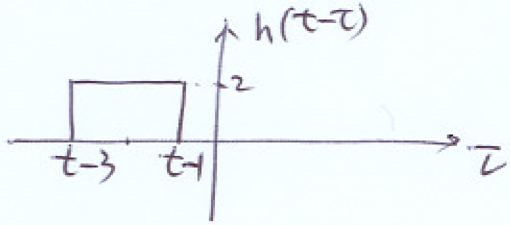


$\alpha=\beta$ ($\beta=1$).



(C) $y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$.



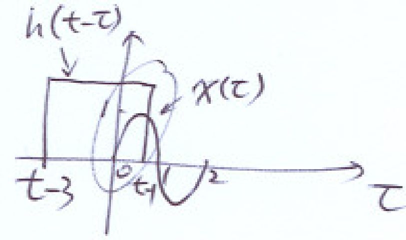


Regions of overlap:

① non-overlap: $t-1<0 \Rightarrow t<1 \Rightarrow y(t)=0$.

② partial-overlap:

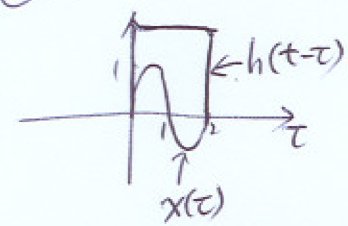
$\begin{cases} t-1>0 \\ t-3<0 \end{cases} \Rightarrow 1<t<3$



$y(t)=\int_0^{t-1} x(\tau)h(t-\tau)\,d\tau$

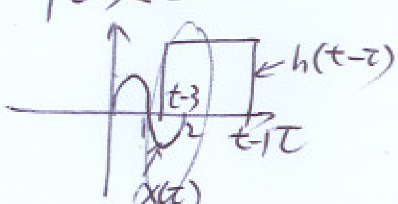
$=\int_0^{t-1}\sin(\pi\tau)\cdot 2\,d\tau=\frac{2}{\pi}\left(1-\cos(\pi(t-1))\right)$.

③ $t-1=2 \Rightarrow t=3$ (full-overlap)



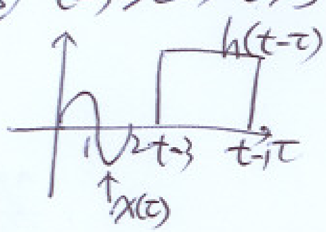
$y(t)=\int_0^{t-1} x(\tau)h(t-\tau)\,d\tau=\int_0^{2}\sin(\pi\tau)\cdot 2\,d\tau=0$.

④ $\begin{cases} t-1>2 \\ t-3<2 \end{cases} \Rightarrow 3<t<5$ (partial-overlap)



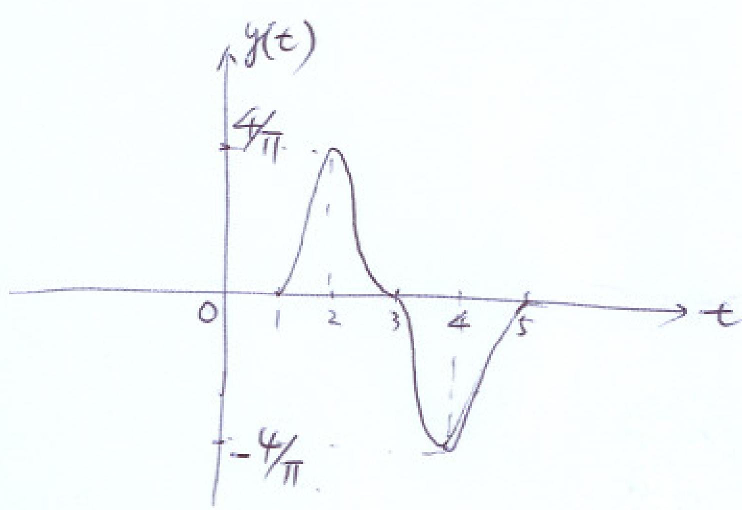
$y(t)=\int_{t-3}^{2} x(\tau)h(t-\tau)\,d\tau=\int_{t-3}^{2}\sin(\pi\tau)\cdot 2\,d\tau=\frac{2}{\pi}\left(\cos(\pi(t-3))-1\right)$

⑤ $t-3>2 \Rightarrow t>5$ (non-overlap)



$y(t)=0$

$$y(t)=\begin{cases} 0, & t<1 \\ \frac{2}{\pi}\left(1-\cos(\pi(t-1))\right), & 1<t<3 \\ \frac{2}{\pi}\left(\cos(\pi(t-3))-1\right), & 3<t<5 \\ 0, & t>5 \end{cases}$$



2.25 (a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(3^k u[-k-1] + \left(\tfrac{1}{3}\right)^k u[k]\right)}_{x[k]} \cdot \underbrace{\left(\tfrac{1}{4}\right)^{n-k} u[n-k+3]}_{h[n-k]}$$

You should learn to do the above calculation without plotting. $\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} 3^k u[-k-1] \cdot \left(\tfrac{1}{4}\right)^{n-k} u[n-k+3] +$

$$\sum_{k=-\infty}^{\infty} \left(\tfrac{1}{3}\right)^k u[k] \cdot \left(\tfrac{1}{4}\right)^{n-k} u[n-k+3]$$

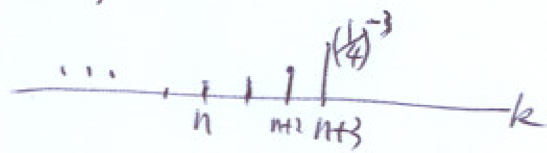
Because $u[-k-1] = \begin{cases} 1, & k \le -1 \\ 0, & k > -1 \end{cases}$ and $u[k] = \begin{cases} 1, & k \ge 0 \\ 0, & k < 0 \end{cases}$

$$\therefore y[n] = \sum_{k=-\infty}^{-1} 3^k \left(\tfrac{1}{4}\right)^{n-k} u[n-k+3] + \sum_{k=0}^{\infty} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^{n-k} u[n-k+3] \quad \circledast$$

$\because u[n-k+3] = \begin{cases} 1, & n+3 \ge k \\ 0, & k > n+3 \end{cases} \Rightarrow$ we determine range of n (essentially regions of overlap)

Range of $x[k]$: ignore the value. Range of $h[n-k]$

both-sided



① $n+3 \le -1, \Rightarrow n \le -4$, $y[n] = \sum_{k=-\infty}^{n+3} 3^k \left(\tfrac{1}{4}\right)^{n-k} \underbrace{u[n-k+3]}_{=1\ (k \le n+3)}$ ← use first term of $\circledast$ only.

$$= \sum_{k=-\infty}^{n+3} \left(\tfrac{1}{4}\right)^n (3 \cdot 4)^k$$

$$= \left(\tfrac{12^4}{11}\right) \cdot 3^n$$

use two terms of $\circledast$

② $n+3 \ge 0 \Rightarrow n \ge -3$, $y[n] = \sum_{k=-\infty}^{-1} 3^k \left(\tfrac{1}{4}\right)^{n-k} \underbrace{u[n-k+3]}_{=1\ (k<0,\ n+3 \ge 0)} + \sum_{k=0}^{n+3} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^{n-k} \underbrace{u[n-k+3]}_{=1\ (k \le n+3)}$

$$= \sum_{k=-\infty}^{-1} 3^k \left(\tfrac{1}{4}\right)^{n-k} + \sum_{k=0}^{n+3} \left(\tfrac{1}{3}\right)^k \left(\tfrac{1}{4}\right)^{n-k}$$

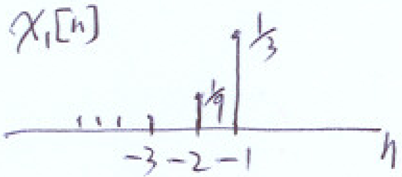
$$= \left(\tfrac{1}{11}\right)\left(\tfrac{1}{4}\right)^n + 3\left(\tfrac{1}{4}\right)^n \left(\left(\tfrac{4}{3}\right)^{n+4} - 1\right) = \left(\tfrac{1}{11}\right)\left(\tfrac{1}{4}\right)^n - 3\left(\tfrac{1}{4}\right)^n + \left(\tfrac{4^4}{3^3}\right)\left(\tfrac{1}{3}\right)^n$$

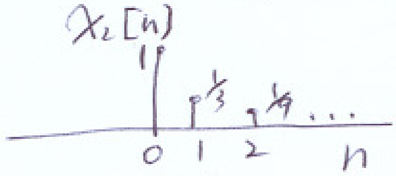
$$\Rightarrow y[n] = \begin{cases} \left(\tfrac{12^4}{11}\right) \cdot 3^n, & n \le -4 \\ \left(\tfrac{1}{11}\right)\left(\tfrac{1}{4}\right)^n - 3\left(\tfrac{1}{4}\right)^n + \left(\tfrac{4^4}{3^3}\right)\left(\tfrac{1}{3}\right)^n, & n \ge -3 \end{cases}$$

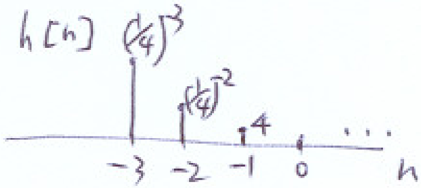
(b) Distributive law:

$$y[n] = (x_1[n] + x_2[n]) * h[n] = \overbrace{x_1[n] * h[n]}^{y_1[n]} + \overbrace{x_2[n] * h[n]}^{y_2[n]}$$

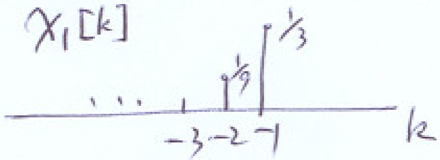
Let $x_1[n] = 3^n u[-n-1]$, $x_2[n] = (\frac{1}{3})^n u[n]$.

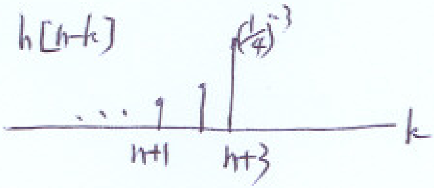






$$y_1[n] = x_1[n] * h[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot h[n-k]$$





Regions of overlap:

① $n+3 \le -1 \Rightarrow n \le -4$, ($x_1[k]$ covers $h[n-k]$)

$$y_1[n] = \sum_{k=-\infty}^{n+3} 3^k \underbrace{u[-k-1]}_{=1\,(k \le n+3 \le -1)} (\tfrac{1}{4})^{n-k} \underbrace{u[n-k+3]}_{=1\,(k \le n+3)}$$

$$= \sum_{k=-\infty}^{n+3} 3^k (\tfrac{1}{4})^{n-k} = \left(\frac{12^4}{11}\right) \cdot 3^n$$

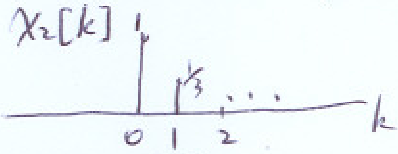
② $n+3 > -1 \Rightarrow n > -4$. ($h[n-k]$ covers $x_1[k]$)

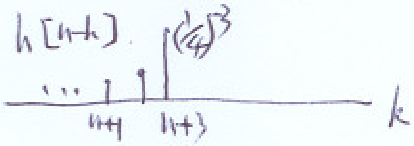
$$y_1[n] = \sum_{k=-\infty}^{-1} 3^k \underbrace{u[-k-1]}_{=1\,(k \le -1)} (\tfrac{1}{4})^{n-k} \underbrace{u[n-k+3]}_{=1\,(k \le -1 < n+3)}$$

$$= \sum_{k=-\infty}^{-1} 3^k (\tfrac{1}{4})^{n-k} = (\tfrac{1}{11})(\tfrac{1}{4})^n$$

$$\therefore y_1[n] = \begin{cases} \left(\frac{12^4}{11}\right) \cdot 3^n, & n \le -4 \\ (\frac{1}{11})(\frac{1}{4})^n, & n > -4 \end{cases}$$

$$y_2[n] = x_2[n] * h[n] = \sum_{k=-\infty}^{\infty} x_2[k] \cdot h[n-k]$$





Regions of overlap.

① $n+3 < 0$: non-overlap: $y_2[n] = 0$.

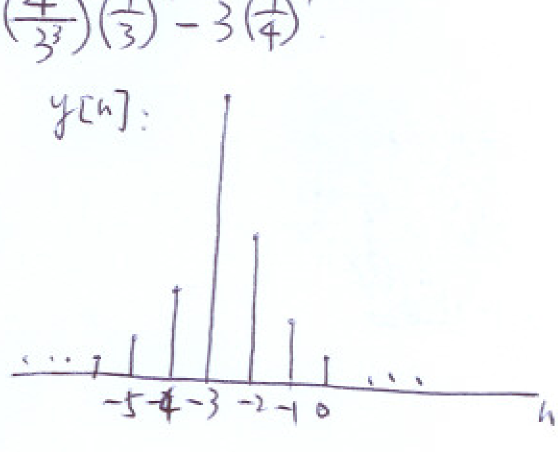
② $n+3 \ge 0$. overlap.

$$y_2[n] = \sum_{k=0}^{n+3} (\tfrac{1}{3})^k \underbrace{u[k]}_{=1\,(k \ge 0)} (\tfrac{1}{4})^{n-k} \underbrace{u[n-k+3]}_{=1\,(k \le n+3)}$$

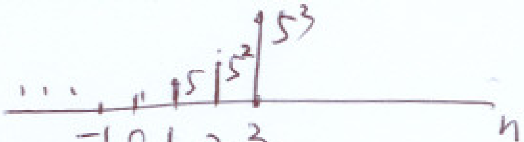
$$= \sum_{k=0}^{n+3} (\tfrac{1}{3})^k (\tfrac{1}{4})^{n-k} = \left(\frac{4^4}{3^3}\right)(\tfrac{1}{3})^n - 3(\tfrac{1}{4})^n$$

$$\therefore y_2[n] = \begin{cases} 0, & n < -3 \\ \left(\frac{4^4}{3^3}\right)(\frac{1}{3})^n - 3(\frac{1}{4})^n, & n \ge -3 \end{cases}$$

$$y[n] = y_1[n] + y_2[n] = \begin{cases} \left(\frac{12^4}{11}\right) \cdot 3^n, & n \le -4 \\ (\frac{1}{11})(\frac{1}{4})^n + \left(\frac{4^4}{3^3}\right)(\frac{1}{3})^n - 3(\frac{1}{4})^n, & n \ge -3. \end{cases}$$



2.28 (d) $h[n] = 5^n u[3-n]$

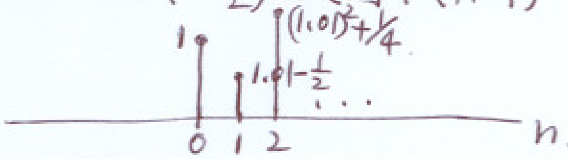


left-sided

$h[n] \neq 0$ for $n<0 \Rightarrow$ not causal

$$\sum_{k=-\infty}^{\infty}|h[k]| = \sum_{k=-\infty}^{3} 5^k = \frac{5^4}{4} < \infty \Rightarrow \text{stable}$$

(e) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[n-1]$



$h[n] = 0$ for $n<0 \Rightarrow$ causal

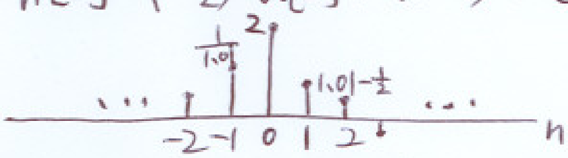
$$\sum_{k=-\infty}^{\infty}|h[k]| = \sum_{k=-\infty}^{\infty}\left|(-\tfrac{1}{2})^k u[k] + (1.01)^k u[k-1]\right|$$

$$= \sum_{k=-\infty}^{\infty}(-\tfrac{1}{2})^k u[k] + (1.01)^k u[k-1] \quad \text{since } (-\tfrac{1}{2})^k u[k] + (1.01)^k u[k-1] \geq 0 \text{ for } k \in (-\infty, \infty).$$

$$= \sum_{k=-\infty}^{\infty}(-\tfrac{1}{2})^k u[k] + \sum_{k=-\infty}^{\infty}(1.01)^k u[k-1]$$

$$= \underbrace{\sum_{k=0}^{\infty}(-\tfrac{1}{2})^k}_{=\frac{2}{3}} + \underbrace{\sum_{k=1}^{\infty}(1.01)^k}_{\text{unbounded}} \rightarrow \infty \Rightarrow \text{unstable}$$

(f) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$



$h[n]$ two-sided

$h[n] \neq 0$ for $n<0 \Rightarrow$ not causal.

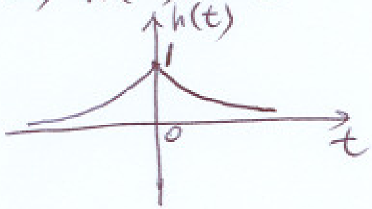
$$\sum_{k=-\infty}^{\infty}|h[k]| = \sum_{k=-\infty}^{\infty}\left|(-\tfrac{1}{2})^k u[k] + (1.01)^k u[1-k]\right|$$

$$= \sum_{k=-\infty}^{-1}\left|(-\tfrac{1}{2})^k u[k] + (1.01)^k u[1-k]\right| + \overset{\text{for } k=0}{2} + \overset{k=1}{1.01-\tfrac{1}{2}} + \sum_{k=2}^{\infty}\left|(-\tfrac{1}{2})^k u[k] + 1.01^k u[1-k]\right|$$

$$= \sum_{k=-\infty}^{-1}(1.01)^k + 2.51 + \sum_{k=2}^{\infty}\left|(-\tfrac{1}{2})^k\right|$$

$$= 100 + 2.51 + \sum_{k=2}^{\infty}(\tfrac{1}{2})^k = 102.51 + 0.5 = 103.01 < \infty \Rightarrow \text{stable}$$

2.29 (e) $h(t)=e^{-6|t|}$



$h(t)\neq 0$ for $t<0 \Rightarrow$ ~~unstable~~ <u>not causal</u>

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau=\int_{-\infty}^{\infty}e^{-6|\tau|}d\tau=\int_{-\infty}^{0}e^{6\tau}d\tau+\int_{0}^{\infty}e^{-6\tau}d\tau$$

$$=1/3 \Rightarrow \underline{\text{stable}}$$

(f) $h(t)=te^{-t}u(t)$

$h(t)=0$ for $t<0$ since $u(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$.

$\Rightarrow$ <u>causal</u>

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau=\int_{-\infty}^{\infty}|\tau e^{-\tau}u(\tau)|d\tau=\int_{0}^{\infty}\tau e^{-\tau}d\tau=1<\infty$$

$\Rightarrow$ <u>stable</u>

(g) $h(t)=(2e^{-t}-e^{(t-100)/100})u(t)$.

$h(t)=0$ for $t<0$ since $u(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$.

$\Rightarrow$ <u>causal</u>

both exponential

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau=\int_{0}^{\infty}|2e^{-\tau}-e^{(\tau-100)/100}|d\tau \qquad \triangle$$

Let t_1 be the point that $e^{(t_1-100)/100}=2e^{-t_1}$, so $\begin{cases}2e^{-t}>e^{(t-100)/100}, & 0<t<t_1\\ 2e^{-t}<e^{(t-100)/100}, & t_1<t\end{cases}$.

$$\triangle=\underbrace{\int_{0}^{t_1}2e^{-\tau}-e^{(\tau-100)/100}d\tau}_{\text{finite because } \tau\in(0,t_1).}+\underbrace{\int_{t_1}^{\infty}e^{(\tau-100)/100}-2e^{-\tau}d\tau}$$

$$=\underbrace{\int_{t_1}^{\infty}e^{(\tau-100)/100}d\tau}_{\text{infinity}}-\underbrace{2\int_{t_1}^{\infty}e^{-\tau}d\tau}_{=2e^{-t_1}}$$

$\Rightarrow \triangle \rightarrow \infty$, <u>unstable</u>.

2.43 (c) $$y[n] = x[n] * (h_1[n] * h_2[n]) \quad \text{(cascade of 2 LTI)}$$
$$= x[n] * (h_2[n] * h_1[n]) \quad \text{(Commutativity)}$$
$$= \underbrace{(x[n] * h_2[n])}_{y_1[n]} * h_1[n] \quad \text{(Associativity)}$$

Now $x[n] = \delta[n] - a\delta[n-1]$,

$h_2[n] = a^n u[n]$, $|a| < 1$.

$$y_1[n] = x[n] * h_2[n]$$
$$= (\delta[n] - a\delta[n-1]) * a^n u[n]$$
$$= \delta[n] * a^n u[n] - a\delta[n-1] * a^n u[n] \quad \text{(distributive law)}$$
$$= a^n u[n] - a \cdot a^{n-1} u[n-1]$$
(since any signal $x[n]$ convolved with $\delta[n-n_0]$ will be $x[n-n_0]$).
$$= a^n u[n] - a^n u[n-1]$$
$$= \delta[n].$$

$$y[n] = y_1[n] * h_1[n]$$
$$= \delta[n] * \sin 8n$$
$$= \sin 8n.$$