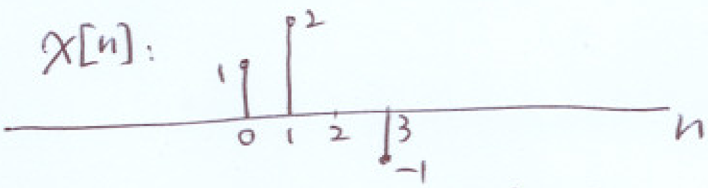
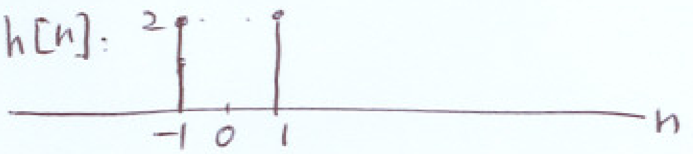
2.1 $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$.

$x[n]$:



$h[n] = 2\delta[n+1] + 2\delta[n-1]$

$h[n]$:



(a) $y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$.

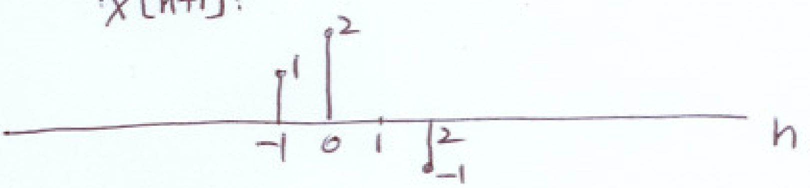
Since $h[k] \neq 0$ when $k = -1$ or 1, so

$y_1[n] = h[-1] \cdot x[n+1] + h[1]\, x[n-1]$.

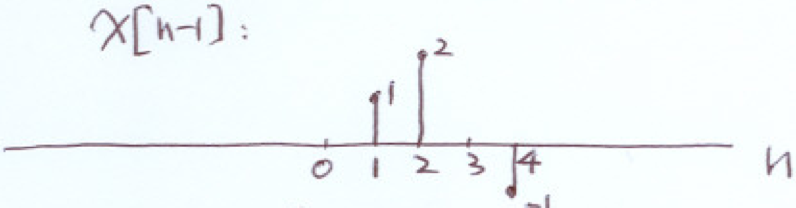
Since $h[-1] = h[1] = 2$,

$y_1[n] = 2x[n+1] + 2x[n-1]$. $\quad (*)$

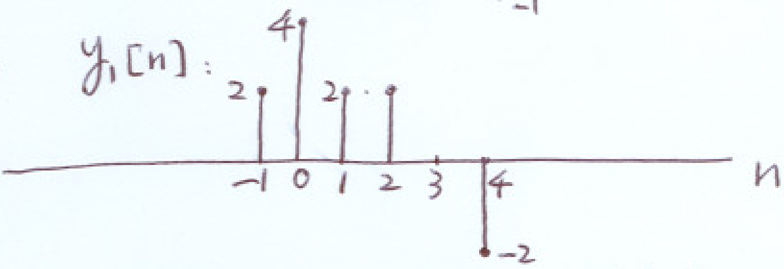
$x[n+1]$:



$x[n-1]$:



$\Rightarrow$ $y_1[n]$:



i.e., $y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$.

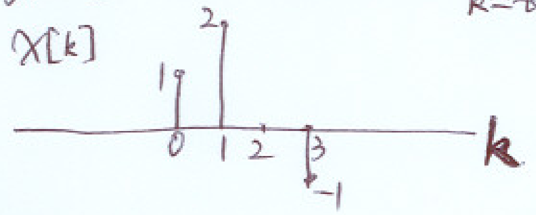
Alternatively, from $(*)$, we know

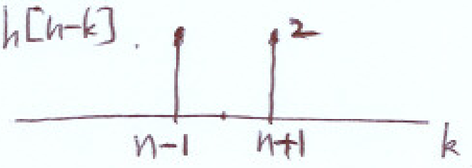
$$\begin{cases} x[n+1] = \delta[n+1] + 2\delta[n] - \delta[n-2] \\ x[n-1] = \delta[n-1] + 2\delta[n-2] - \delta[n-4]. \end{cases}$$

$$\Rightarrow y_1[n] = 2(x[n+1] + x[n-1])$$
$$= 2(\delta[n+1] + 2\delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-4])$$
$$= 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4].$$

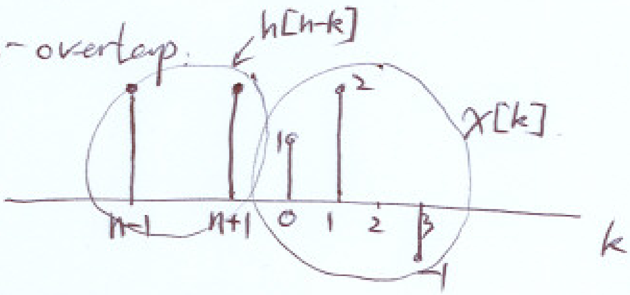
5 regions of overlap:

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k].$$





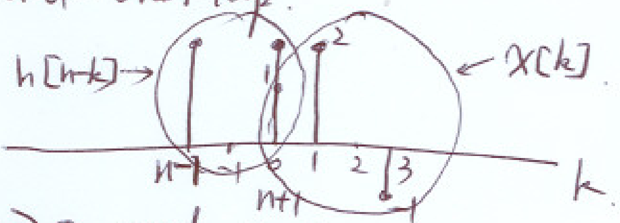
Region 1: non-overlap.



$n+1 < 0 \Leftrightarrow n < -1.$

$y_1[n] = 0$

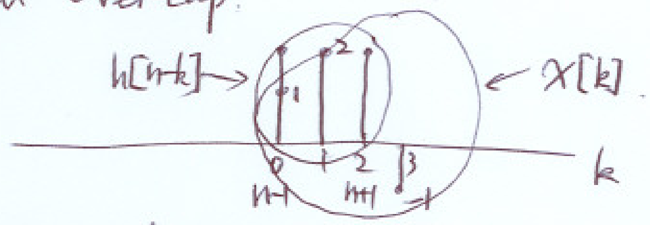
Region 2: partial-overlap.



$n+1 \geq 0$ and $n-1 < 0$

$\Rightarrow -1 \leq n < 1, \quad y_1[-1] = 2, \ y_1[0] = 4.$

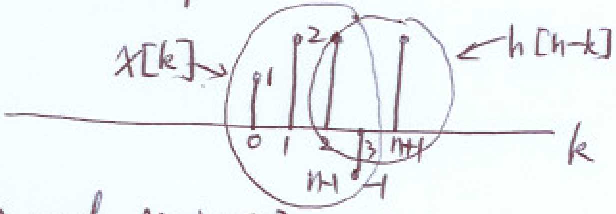
Region 3: full-overlap.



$n-1 \geq 0$ and $n+1 \leq 3.$

$\Rightarrow 1 \leq n \leq 2, \ y_1[1] = 2, \ y_1[2] = 2.$

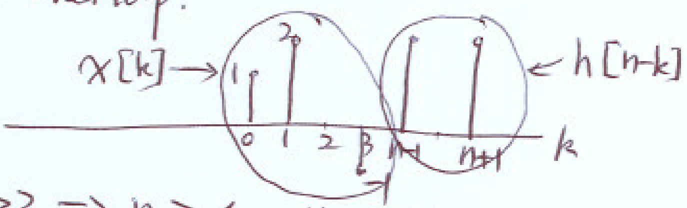
Region 4: partial-overlap:



$n+1 > 3$ and $n-1 \leq 3$

$\Rightarrow 2 < n \leq 4, \quad y_1[3] = 0, \ y_1[4] = -2.$

Region 5: non-overlap.



$n-1 > 3 \Rightarrow n > 4, \ y_1[n] = 0$

(b) We know $y_2[n] = x[n+2] * h[n]$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n+2-k].$$

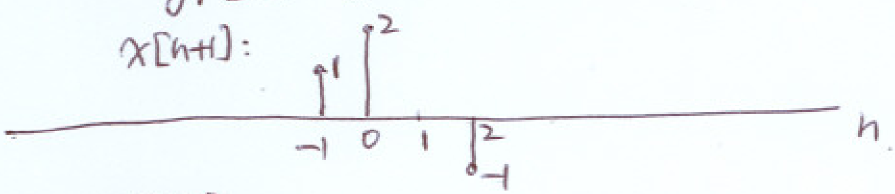
Since $h[k] \neq 0$ when $k = -1$ or 1,

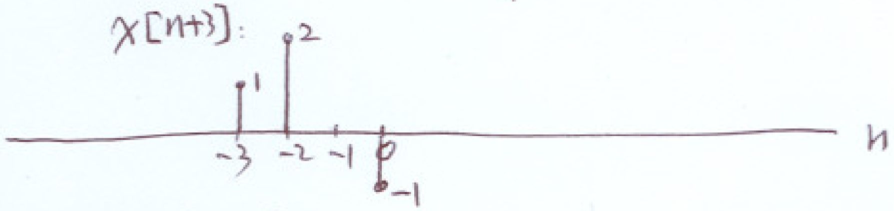
$$y_2[n] = h[-1] \cdot x[n+2-(-1)] + h[1] \cdot x[n+2-1]$$
$$= h[-1] \cdot x[n+3] + h[1] \cdot x[n+1].$$
$$= 2 \cdot x[n+3] + 2x[n+1].$$

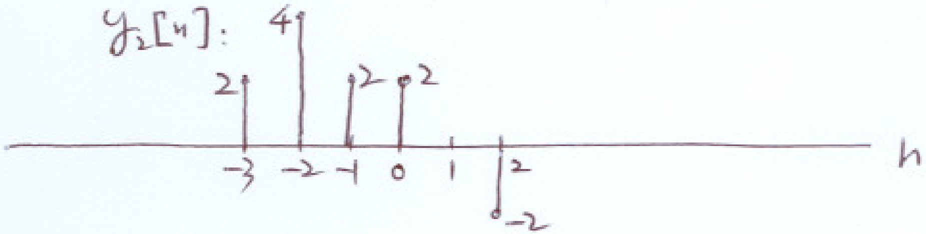
Now, $x[n+1] = \delta[n+1] + 2\delta[n] - \delta[n-2].$

$x[n+3] = \delta[n+3] + 2\delta[n+2] - \delta[n].$

$$\Rightarrow y_2[n] = 2(x[n+3] + x[n+1])$$
$$= 2(\delta[n+3] + 2\delta[n+2] + \delta[n+1] + \delta[n] - \delta[n-2])$$
$$= 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2].$$
$$= y_1[n+2].$$

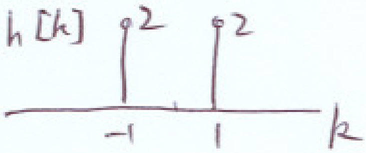


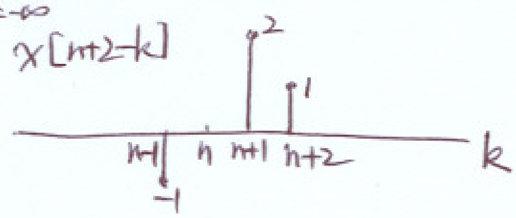




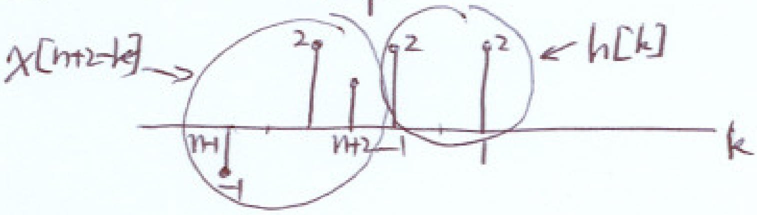
5 regions of overlap:

$$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n+2-k].$$



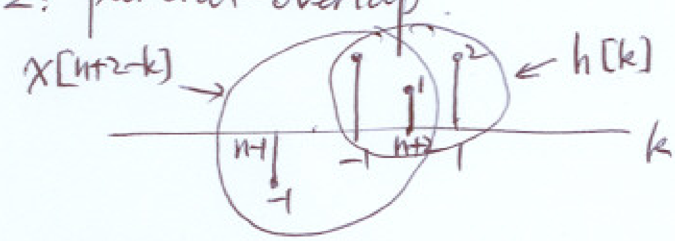


Region 1: non-overlap.



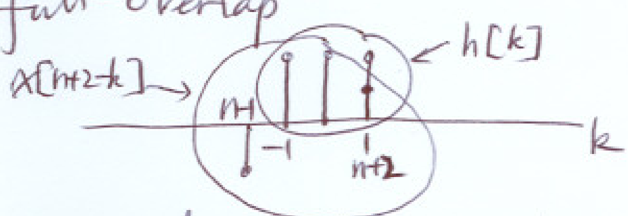
$n+2 < -1 \Rightarrow n < -3$, $y_2[n] = 0$.

Region 2: partial-overlap



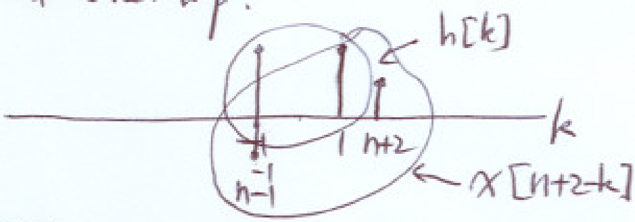
$n+2 \geq -1$ and $n+2 < 1 \Rightarrow -3 \leq n < -1$, $y_2[-3] = 2$, $y_2[-2] = 4$.

Region 3: full-overlap



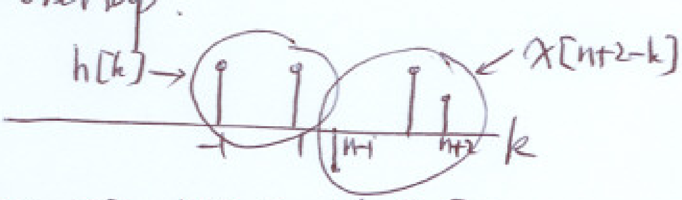
$n+2 \geq 1$ and $n-1 \leq -1 \Rightarrow -1 \leq n \leq 0$, $y_2[-1] = 2$, $y_2[0] = 2$.

Region 4: partial-overlap.



~~n+2~~ $n-1 \leq 1$ and $n-1 > -1 \Rightarrow 0 < n \leq 2$, $y_2[1] = 0$, $y_2[2] = -2$.

Region 5: non-overlap.



$n-1 > 1 \Rightarrow n > 2$, $y_2[n] = 0$

(c) $y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[\underline{n+2}-k]$.

$x[k] \neq 0$ when $k = 0, 1, 3$.

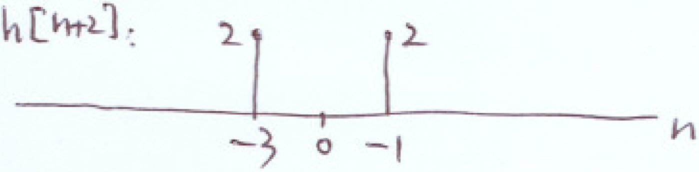
$\Rightarrow y_3[n] = x[0] \cdot h[n+2-0] + x[1] \cdot h[n+2-1] + x[3] \cdot h[n+2-3]$

$= x[0] \cdot h[n+2] + x[1] \cdot h[n+1] + x[3] \cdot h[n-1]$.

$= 1 \cdot h[n+2] + 2 \cdot h[n+1] - 1 \cdot h[n-1]$.

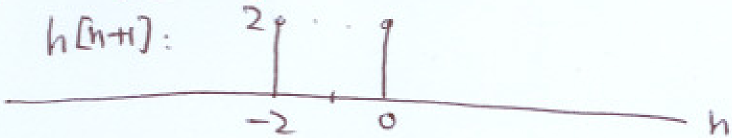
$h[n+2] = 2\delta[n+2+1] + 2\delta[n+2-1] = 2\delta[n+3] + 2\delta[n+1]$.

$h[n+2]$:



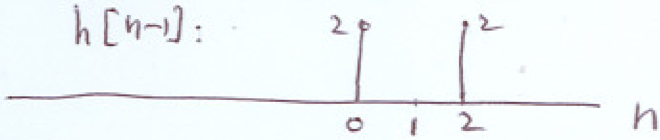
$h[n+1] = 2\delta[n+2] + 2\delta[n]$.

$h[n+1]$:



$h[n-1] = 2\delta[n] + 2\delta[n-2]$.

$h[n-1]$:

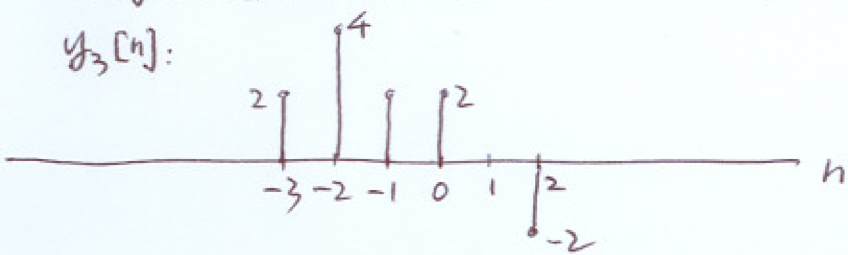


$\Rightarrow y_3[n] = h[n+2] + 2h[n+1] - h[n-1]$

$= 2\delta[n+3] + 2\delta[n+1] + 2(2\delta[n+2] + 2\delta[n]) - (2\delta[n] + 2\delta[n-2])$

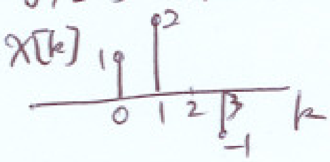
$= 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2] = y_2[n]$.

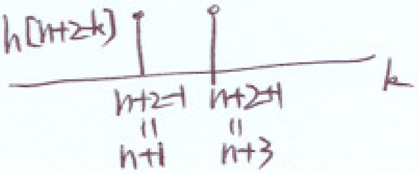
$y_3[n]$:



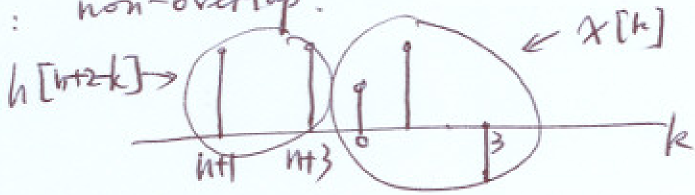
5 regions of overlap:

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n+2-k].$$



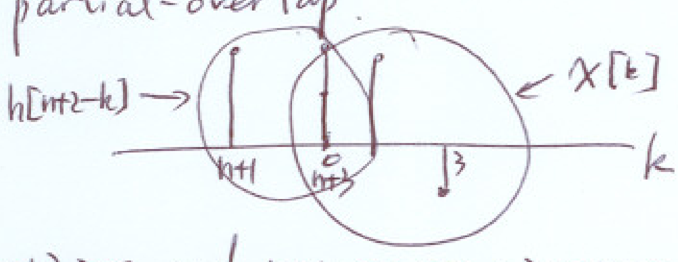


Region 1: non-overlap.



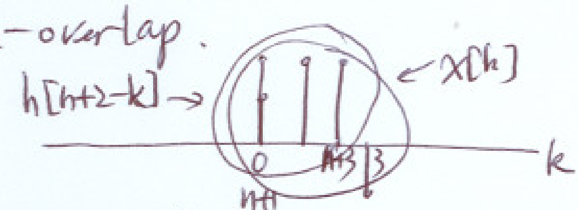
$n+3 < 0 \Rightarrow n < -3$, $y_3[n] = 0$

Region 2: partial-overlap.



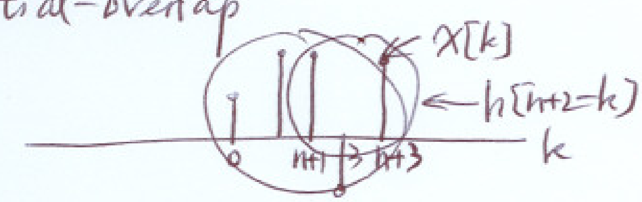
$n+3 \geq 0$ and $n+1 < 0 \Rightarrow -3 \leq n < -1$, $y_3[-3] = 2$, $y_3[-2] = 4$.

Region 3: full-overlap.



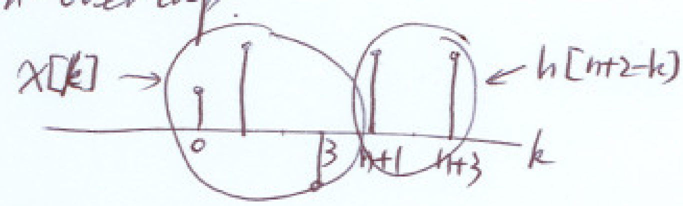
$n+1 \geq 0$ and $n+3 \leq 3 \Rightarrow -1 \leq n \leq 0$. $y_3[-1] = 2$, $y_3[0] = 2$.

Region 4: partial-overlap



$n+1 \leq 3$ and $n+3 > 3 \Rightarrow 0 < n \leq 2$, $y_3[1] = 0$, $y_3[2] = -2$.

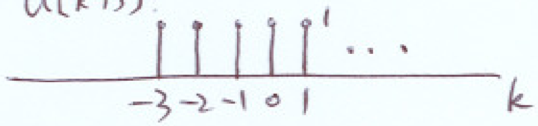
Region 5: non-overlap.



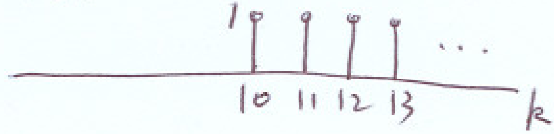
$n+1 > 3 \Rightarrow n > 2$, $y_3[n] = 0$.

2.2. $h[k] = \left(\frac{1}{2}\right)^{k-1} \{u[k+3] - u[k-10]\}$.

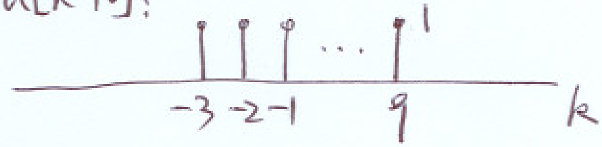
$u[k+3]$:



$u[k-10]$:



$\Rightarrow$ $u[k+3] - u[k-10]$:



So $h[k] \neq 0$ for $-3 \leq k \leq 9$.

$h[-k]$ is the flipped version of $h[k]$, and

$h[-k] \neq 0$ when $-9 \leq k \leq 3$.

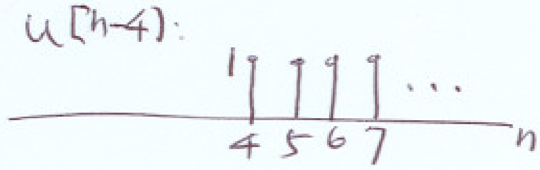
If we shift $h[-k]$ by n units to get $h[n-k]$,

$\begin{cases} \text{When } n>0, \text{ shift to right}, & h[n-k] \neq 0 \text{ when } -9+n \leq k \leq 3+n. \\ \text{When } n<0, \text{ shift to left}, & h[n-k] \neq 0 \text{ when } -9+n \leq k \leq 3+n. \end{cases}$

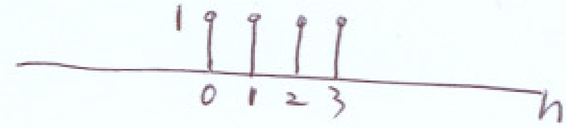
Remember here, from $h[k]$ to $h[n-k]$, we do time reversal first and shift next, be careful with the direction of shift.

$\Rightarrow A = n-9,\ B = n+3$.

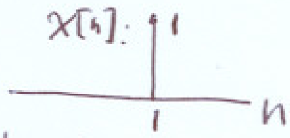
2.7 $u[n-4]$:



$\Rightarrow$ $g[n]$:



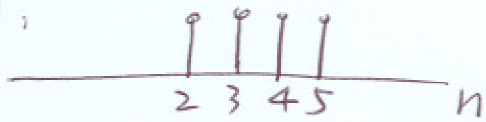
(a) When $x[n] = \delta[n-1]$ $x[n]$:



Since $x[k] \neq 0$ when $k=1$,

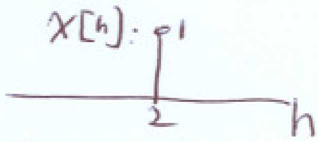
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, g[n-2k] = x[1]\, g[n-2\cdot 1] = g[n-2].$$

$y[n] = g[n-2]$:



or $g[n-2] = u[n-2] - u[n-6] = y[n]$.

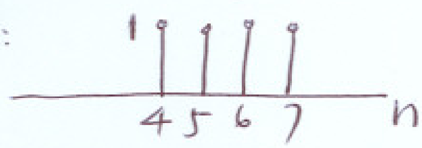
(b) When $x[n] = \delta[n-2]$ $x[n]$:



Since $x[k] \neq 0$, when $k=2$,

$$y[n] = x[2] \cdot g[n-2\cdot 2] = g[n-4].$$

$y[n] = g[n-4]$:



or $g[n-4] = y[n] = u[n-4] - u[n-8]$.

(c) Since input in (b) is the shifted version of input in (a) by 1 to right, suppose "S" is time-invariant, the output $y[n]$ in (b) should also be the shifted version of output $y[n]$ in (a) by 1 to the right.

Now, this is not the case. So "S" is not LTI.

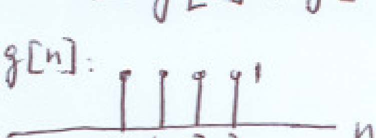
(d) When $x[n] = u[n]$, ($u[n] = 1$ when $n \geq 0$, and $u[n] = 0$ if $n < 0$)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, g[n-2k]$$

$$= \sum_{k=0}^{\infty} x[k]\, g[n-2k] = \sum_{k=0}^{\infty} u[k]\, g[n-2k] = \sum_{k=0}^{\infty} g[n-2k].$$

$$= g[n] + g[n-2] + g[n-4] + \cdots$$

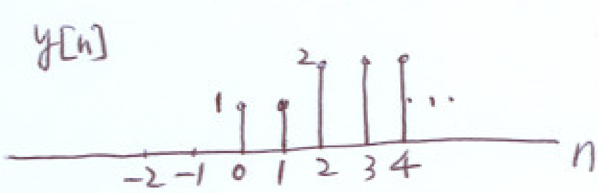
$g[n]$:



So, $y[n] = \begin{cases} 1, & n = 0 \text{ or } 1, \\ 2, & n > 1 \\ 0, & n < 0 \end{cases}$

$g[n-2]$:

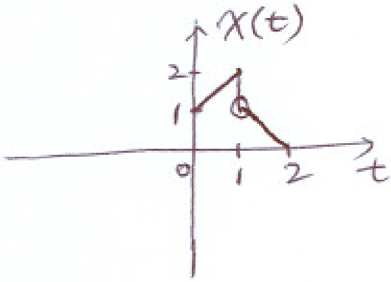
$y[n]$



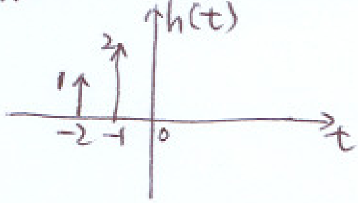
$g[n-4]$:

$$\text{or, } y[n] = \sum_{k=0}^{\infty} g[n-2k]$$
$$= g[n] + g[n-2] + g[n-4] + \dots$$
$$= (u[n] - \cancel{u[n-4]}) + (u[n-2] - u[n-6]) + (\cancel{u[n-4]} - u[n-8]) + \dots$$
$$= u[n] + u[n-2] \quad \text{(After cancelling terms).}$$

2.8 $x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{o.w.} \end{cases}$



$h(t) = \delta(t+2) + 2\delta(t+1)$.

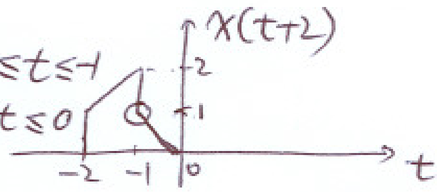


Now, $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$.

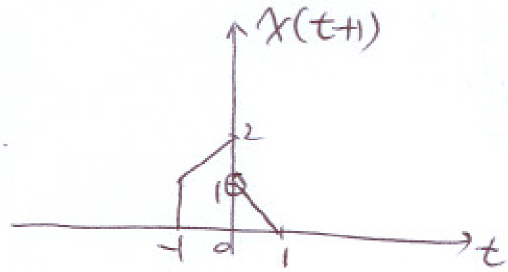
Sine $h(\tau) \neq 0$ when $\tau = -1$ and -2.

$$x(t) * h(t) = \int_{-\infty}^{\infty} (\delta(\tau+2) + 2\delta(\tau+1)) x(t-\tau) d\tau$$
$$= \int_{-\infty}^{\infty} \delta(\tau+2) x(t-\tau) d\tau + \int_{-\infty}^{\infty} 2\delta(\tau+1) x(t-\tau) d\tau$$
$$= \int_{-\infty}^{\infty} \delta(\tau+2) x(t+2) d\tau + \int_{-\infty}^{\infty} 2\delta(\tau+1) x(t+1) d\tau$$
$$= x(t+2) \int_{-\infty}^{\infty} \delta(\tau+2) d\tau + 2x(t+1) \int_{-\infty}^{\infty} \delta(\tau+1) d\tau$$
$$= x(t+2) + 2x(t+1)$$

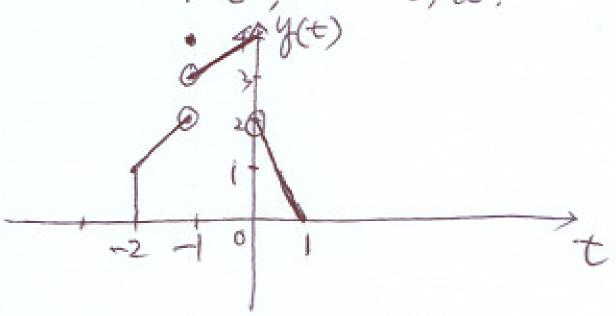
$x(t+2) = \begin{cases} t+3, & 0 \le t+2 \le 1 \Leftrightarrow -2 \le t \le -1 \\ -t, & 1 < t+2 \le 2 \Leftrightarrow -1 < t \le 0 \\ 0, & \text{o.w.} \end{cases}$



$x(t+1) = \begin{cases} t+2, & 0 \le t+1 \le 1 \Leftrightarrow -1 \le t \le 0 \\ 1-t, & 1 < t+1 \le 2 \Leftrightarrow 0 < t \le 1. \\ 0, & \text{o.w.} \end{cases}$



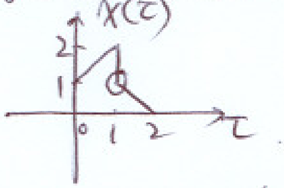
$$\Rightarrow \quad y(t)=\begin{cases} t+3, & -2\le t<-1 \\ -t+2(t+2)=t+4, & -1<t\le 0 \\ 2-2t, & 0<t\le 1 \\ 0, & \text{o.w.} \end{cases} \qquad \left\{ 4, \quad t=-1 \right.$$

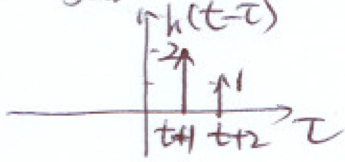


Be careful with the boundary points...

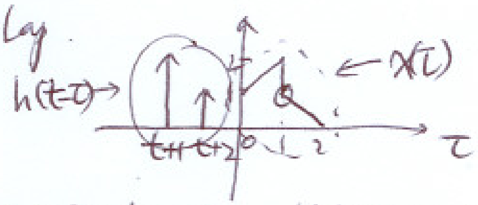
5 regions of overlap:

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$$



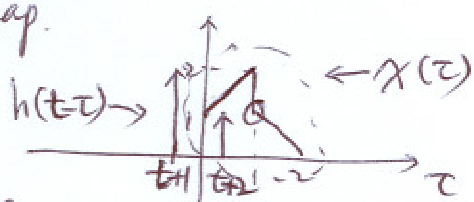


R1: non-overlap



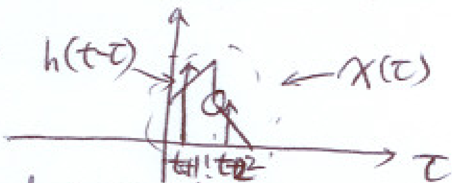
$t+2<0 \Rightarrow t<2$, $\underline{y(t)=0}$

R2: partial overlap



$t+2\ge 0$ and $t+1<0 \Rightarrow -2\le t<-1$, $\underline{y(t)=t+2+1=t+3}$.

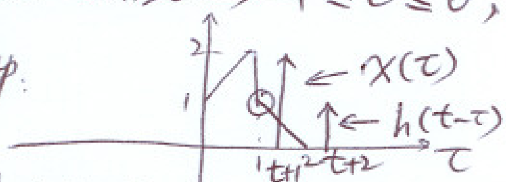
R3: full-overlap.



$t+2\le 2$ and $t+1\ge 0 \Rightarrow -1\le t\le 0$, when $t=-1$, $\underline{y(t)=4}$,

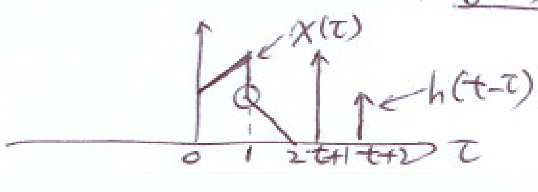
$-1<t\le 0$, $\underline{y(t)=2(t+2)-t = t+4}$.

R4: partial-overlap.



$t+2>2$ and $t+1\le 2 \Rightarrow 0<t\le 1$, $\underline{y(t)=2(2-(t+1))=2-2t}$

R5: non-overlap:



$\underline{t+1>2 \Rightarrow t>1,\ y(t)=0}$